Van Aubel’s Theorem in the Einstein Relativistic Velocity Model of Hyperbolic Geometry

Cătălin Barbu

“Vasile Alecsandri” National College - Bacău, str. Vasile Alecsandri, nr. 37, 600011, Bacău, Romania
E-mail: kafka.mate@yahoo.com

In this note, we present a proof to the Van Aubel Theorem in the Einstein Relativistic Velocity Model of Hyperbolic Geometry.

1 Introduction

Hyperbolic Geometry appeared in the first half of the 19th century as an attempt to understand Euclid’s axiomatic basis of Geometry. It is also known as a type of non-Euclidean Geometry, being in many respects similar to Euclidean Geometry. Hyperbolic Geometry includes similar concepts as distance and angle. Both these geometries have many results in common but many are different. There are known many models for Hyperbolic Geometry, such as: Poincaré disc model, Poincaré half-plane, Klein model, Einstein relativistic velocity model, etc. Here, in this study, we give hyperbolic version of Van Aubel theorem. The well-known Van Aubel theorem states that if $ABC$ is a triangle and $AD, BE, CF$ are concurrent cevians at $P$, then $\frac{AP}{PF} = \frac{AB}{BF} + \frac{AC}{EC}$ (see [1, p 271]).

Let $D$ denote the complex unit disc in complex $z$-plane, i.e.

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$ 

The most general Möbius transformation of $D$ is

$$z \rightarrow \frac{z_0 + z}{1 + \overline{z_0}z} = \frac{e^{i\theta} z_0 + z}{1 + \overline{z_0}z},$$

which induces the Möbius addition $\oplus$ in $D$, allowing the Möbius transformation of the disc to be viewed as a Möbius left gyrotranslation

$$z \rightarrow z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0}z}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $z, z_0 \in D$, and $\overline{z_0}$ is the complex conjugate of $z_0$. Let $\text{Aut}(D, \oplus)$ be the automorphism group of the grupoid $(D, \oplus)$. If we define

$$\text{gyr} : D \times D \rightarrow \text{Aut}(D, \oplus), \text{gyr}[a, b] = a \oplus b \ominus a = \frac{a + b}{1 + \overline{a}b},$$

then is true gyrocommutative law

$$a \ominus b = \text{gyr}[a, b](b \ominus a).$$

A gyrovector space $(G, \oplus, \otimes)$ is a gyrocommutative gyrogroup $(G, \oplus)$ that obeys the following axioms:

1. $\text{gyr}[u, v]a - \text{gyr}[u, v]b = a - b$ for all points $a, b, u, v \in G$.

2. $G$ admits a scalar multiplication, $\oplus$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $a \in G$:

   \begin{align*}
   (G1) \quad & 1 \otimes a = a \\
   (G2) \quad & (r_1 + r_2) \otimes a = r_1 \otimes a \oplus r_2 \otimes a \\
   (G3) \quad & (r_1r_2) \otimes a = r_1 \otimes (r_2 \otimes a) \\
   (G4) \quad & |r| \otimes a = \frac{a}{||r||} \\
   (G5) \quad & \text{gyr}[u, v](r \otimes a) = r \otimes \text{gyr}[u, v]a \\
   (G6) \quad & \text{gyr}[r_1 \otimes v, r_1 \otimes v] = 1 \\
   (G7) \quad & |r| \otimes a = |r| \otimes |a| \\
   (G8) \quad & |a \otimes b| \leq ||a|| \otimes ||b||
   \end{align*}

3. Real vector space structure $(||G||, \oplus, \otimes)$ for the set $||G||$ of onedimensional “vectors”

$$||G|| = \{\pm ||a|| : a \in G\} \subset \mathbb{R}$$

with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $a, b \in G$,

\begin{align*}
(G7) \quad & |r| \otimes a = |r| \otimes |a| \\
(G8) \quad & ||a \otimes b|| \leq ||a|| \otimes ||b||
\end{align*}

Definition 1. Let $ABC$ be a gyrotriangle with sides $a, b, c$ in an Einstein gyrovector space $(V, \oplus, \otimes)$, and let $h_a, h_b, h_c$ be three altitudes of $ABC$ drawn from vertices $A, B, C$ perpendicular to their opposite sides $a, b, c$ or their extension, respectively. The number

$$S_{ABC} = \gamma_a \gamma_b h_a = \gamma_b \gamma_c h_b = \gamma_c \gamma_a h_c$$

is called the gyrotriangle constant of gyrotriangle $ABC$ (here $\gamma = \frac{1}{\sqrt{1 - ||v||^2}}$ is the gamma factor).

(See [2, p 558].)
Theorem 1. (The Gyrotriangle Constant Principle)
Let $A_1BC$ and $A_2BC$ be two gyrotriangles in a Einstein gyrovector plane $(\mathbb{R}_2, \oplus, \otimes)$. $A_1 \neq A_2$ such that the two gyrosegments $A_1A_2$ and $BC$, or their extensions, intersect at a point $P \in \mathbb{R}_2$. Then,
$$
\frac{\gamma_{A_1P}}{\gamma_{A_2P}} = \frac{S_{A_1BC}}{S_{A_2BC}}.
$$
(See [2, p 563].)

Theorem 2. (The Hyperbolic Theorem of Menelaus in Einstein Gyrovector Space)
Let $a_1, a_2,$ and $a_3$ be three non-gyrocollinear points in an Einstein gyrovector space $(V, \oplus, \otimes)$. If a gyroline meets the sides of gyrotriangle $a_1a_2a_3$ at points $a_1, a_2, a_3$, then
$$
\gamma_{a_1a_2} \parallel a_1 a_2 \parallel a_2 a_3 = 1
$$
(See [2, p 453].)

Theorem 3. (The Gyrotriangle Bisector Theorem)
Let $ABC$ be a gyrotriangle in an Einstein gyrovector space $(V, \oplus, \otimes)$, and let $P$ be a point lying on side $BC$ of the gyrotriangle such that $AP$ is a bisector of gyroangle $\angle ABC$. Then,
$$
\gamma_{AP}[AP] \parallel \gamma_{PC}[PC] = \gamma_{AB}[AB] + \gamma_{AC}[AC].
$$
(See [3, p 150].) For further details we refer to the recent book of A. Ungar [2].

2 Main results
In this section, we prove Van Aubel’s theorem in hyperbolic geometry.

Theorem 4. If the point $P$ lies on any side of the hyperbolic triangle $ABC$, and $BC$ meets $AP$ in $D$, $CA$ meets $BP$ in $E$, and $AB$ meets $CP$ in $F$, then
$$
\gamma_{AP} \parallel \gamma_{PD}[PD] = \frac{\gamma_{BC}[BC]}{2} \cdot \frac{\gamma_{AB}[AE]}{\gamma_{EC}[EC]} \cdot \frac{1}{\gamma_{BD}[BD]} + \frac{\gamma_{BC}[BC]}{2} \cdot \frac{\gamma_{FA}[FA]}{\gamma_{FB}[FB]} \cdot \frac{1}{\gamma_{CD}[CD]}.
$$
Proof. If we use the Menelaus’ theorem in the $h$-triangles $ADC$ and $ABD$ for the $h$-lines $BPE$, and $CPF$ respectively, then
$$
\gamma_{AP} \parallel \gamma_{PD}[PD] = \frac{\gamma_{AE}[AE]}{\gamma_{EC}[EC]} \cdot \frac{\gamma_{AB}[BC]}{\gamma_{BD}[BD]} + \frac{\gamma_{BP}[BP]}{\gamma_{PC}[PC]}
$$
and
$$
\gamma_{AP} \parallel \gamma_{PD}[PD] = \frac{\gamma_{FB}[FB]}{\gamma_{FA}[FA]} \cdot \frac{\gamma_{AB}[BC]}{\gamma_{CD}[CD]},
$$
From (1) and (2), we have
$$
2 \cdot \frac{\gamma_{AP}[AP]}{\gamma_{PD}[PD]} = \frac{\gamma_{AE}[AE]}{\gamma_{EC}[EC]} \cdot \frac{\gamma_{AB}[BC]}{\gamma_{BD}[BD]} + \frac{\gamma_{FB}[FB]}{\gamma_{FA}[FA]} \cdot \frac{\gamma_{AB}[BC]}{\gamma_{CD}[CD]},
$$
the conclusion follows. □

Corollary 1. Let $G$ be the centroid of the hyperbolic triangle $ABC$, and $D, E, F$ are the midpoints of hyperbolic sides $BC, CA, AC$ respectively, then
$$
\frac{\gamma_{AG}[AG]}{\gamma_{GD}[GD]} = \frac{\gamma_{BC}[BC]}{2} \cdot \frac{\gamma_{AB}[AE]}{\gamma_{EC}[EC]} \cdot \frac{1}{\gamma_{BD}[BD]} + \frac{\gamma_{BC}[BC]}{2} \cdot \frac{\gamma_{FA}[FA]}{\gamma_{FB}[FB]} \cdot \frac{1}{\gamma_{CD}[CD]}.
$$
Proof. If we use a theorem 5 for the triangle $ABC$ and the centroid $G$, we have
$$
\frac{\gamma_{AG}[AG]}{\gamma_{GD}[GD]} = \frac{\gamma_{BC}[BC]}{2} \cdot \frac{\gamma_{AB}[AE]}{\gamma_{EC}[EC]} \cdot \frac{1}{\gamma_{BD}[BD]} + \frac{\gamma_{BC}[BC]}{2} \cdot \frac{\gamma_{FA}[FA]}{\gamma_{FB}[FB]} \cdot \frac{1}{\gamma_{CD}[CD]},
$$
the conclusion follows. □

Corollary 2. Let $I$ be the incentre of the hyperbolic triangle $ABC$, and let the angle bisectors be $AD, BE, CF$. Then,
$$
\frac{\gamma_{AI}[AI]}{\gamma_{ID}[ID]} = \frac{\gamma_{AB}[AB]}{\gamma_{BC}[BC]} \cdot \frac{\gamma_{AC}[AC]}{\gamma_{BD}[BD]} + \frac{\gamma_{AI}[AI]}{\gamma_{ID}[ID]} \cdot \frac{\gamma_{AB}[AB]}{\gamma_{EC}[EC]} \cdot \frac{\gamma_{AC}[AC]}{\gamma_{BD}[BD]}.
$$
Proof. If we use theorem 4 for the triangle $ABC$, we have
$$
\frac{\gamma_{AB}[AE]}{\gamma_{EC}[EC]} = \frac{\gamma_{AB}[AE]}{\gamma_{BC}[BC]} \cdot \frac{\gamma_{AC}[AC]}{\gamma_{BD}[BD]},
$$
and
$$
\frac{\gamma_{AI}[AI]}{\gamma_{ID}[ID]} = \frac{\gamma_{AB}[AB]}{\gamma_{BC}[BC]} \cdot \frac{\gamma_{AC}[AC]}{\gamma_{BD}[BD]}.
$$
From (5) and (6), we have
$$
\frac{\gamma_{AI}[AI]}{\gamma_{ID}[ID]} = \frac{\gamma_{AB}[AB]}{\gamma_{BC}[BC]} \cdot \frac{\gamma_{AC}[AC]}{\gamma_{BD}[BD]} + \frac{\gamma_{AI}[AI]}{\gamma_{ID}[ID]} \cdot \frac{\gamma_{AB}[AB]}{\gamma_{EC}[EC]} \cdot \frac{\gamma_{AC}[AC]}{\gamma_{BD}[BD]},
$$
the conclusion follows. □
The Einstein relativistic velocity model is another model of hyperbolic geometry. Many of the theorems of Euclidean geometry are relatively similar form in the Einstein relativistic velocity model. Aubel’s theorem for gyrotriangle is an example in this respect. In the Euclidean limit of large $s$, $s \rightarrow \infty$, gamma factor $\gamma_v$ reduces to 1, so that the gyroequality (1) reduces to the

$$\frac{|AP|}{|PD|} = \frac{|BC|}{2} \left( \frac{|AE|}{|EC|} \cdot \frac{1}{|BD|} \cdot \frac{|FA|}{|FB|} \cdot \frac{1}{|CD|} \right)$$

in Euclidean geometry. We observe that the previous equality is an equivalent form to the Van Aubel’s theorem of euclidean geometry.

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References