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Pappus's harmonic theorem in the Einstein relativistic velocity model of hyperbolic geometry

Laurian-Ioan Pişcoran and Cătălin Barbu

Abstract. In this note, we present a proof of Pappus's harmonic theorem in the Einstein relativistic velocity model of hyperbolic geometry.

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1. Introduction

Hyperbolic geometry appeared in the first half of the 19^{th} century as an attempt to understand Euclid's axiomatic basis for geometry. It is also known as a type of non-Euclidean geometry, being in many respects similar to Euclidean geometry. Hyperbolic geometry includes such concepts as: distance, angle and both of them have many theorems in common. There are known many main models for hyperbolic geometry, such as: Poincaré disc model, Poincaré half-plane, Klein model, Einstein relativistic velocity model, etc. The hyperbolic geometry is a non-Euclidian geometry. Here, in this study, we present a proof of Pappus's harmonic theorem in the Einstein relativistic velocity model of hyperbolic geometry. Pappus's harmonic theorem states that if $A'B'C'$ is the cevian triangle of point M with respect to the triangle ABC such that the lines $B'C'$ and BC meet at A'', then $\frac{A''B}{A''C} = \frac{A'B}{A'C}$ [4].

Let D denote the complex unit disc in complex z - plane, i.e.

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

The most general Möbius transformation of D is

$$z \to e^{i\theta}\frac{z_0+z}{1+\overline{z_0}z} = e^{i\theta}(z_0 \oplus z),$$

which induces the Möbius addition $\oplus$ in D, allowing the Möbius transformation of the disc to be viewed as a Möbius left gyrotranslation

$$z \to z_0 \oplus z = \frac{z_0+z}{1+\overline{z_0}z}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $z, z_0 \in D$, and $\overline{z_0}$ is the complex conjugate of z_0. Let $Aut(D,\oplus)$ be the automorphism group of the grupoid $(D,\oplus)$. If we define

$$gyr : D \times D \to Aut(D,\oplus), gyr[a,b] = \frac{a \oplus b}{b \oplus a} = \frac{1+a\overline{b}}{1+\overline{a}b},$$

then is true gyrocommutative law

$$a \oplus b = gyr[a,b](b \oplus a).$$

A gyrovector space $(G,\oplus,\otimes)$ is a gyrocommutative gyrogroup $(G,\oplus)$ that obeys the following axioms:

(1) $gyr[\mathbf{u},\mathbf{v}]\mathbf{a}\cdot\, gyr[\mathbf{u},\mathbf{v}]\mathbf{b} = \mathbf{a}\cdot\mathbf{b}$ for all points $\mathbf{a},\mathbf{b},\mathbf{u},\mathbf{v} \in G$.

(2) G admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $\mathbf{a} \in G$:

(G1) $1 \otimes \mathbf{a} = \mathbf{a}$

(G2) $(r_1+r_2)\otimes\mathbf{a} = r_1\otimes\mathbf{a} \oplus r_2\otimes\mathbf{a}$

(G3) $(r_1r_2)\otimes\mathbf{a} = r_1\otimes(r_2\otimes\mathbf{a})$

(G4) $\frac{|r|\otimes\mathbf{a}}{\|r\otimes\mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$

(G5) $gyr[\mathbf{u},\mathbf{v}](r\otimes\mathbf{a}) = r\otimes gyr[\mathbf{u},\mathbf{v}]\mathbf{a}$

(G6) $gyr[r_1\otimes\mathbf{v}, r_1\otimes\mathbf{v}] = 1$

(3) Real vector space structure $(\|G\|,\oplus,\otimes)$ for the set $\|G\|$ of onedimensional "vectors"

$$\|G\| = \{\pm\|\mathbf{a}\| : \mathbf{a} \in G\} \subset \mathbb{R}$$

with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\mathbf{a},\mathbf{b} \in G$,

(G7) $\|r\otimes\mathbf{a}\| = |r|\otimes\|\mathbf{a}\|$

(G8) $\|\mathbf{a}\oplus\mathbf{b}\| \le \|\mathbf{a}\| \oplus \|\mathbf{b}\|$

Theorem 1.1. (The Hyperbolic Theorem of Ceva in Einstein Gyrovector Space). *Let $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a}_3$ be three non-gyrocollinear points in an Einstein gyrovector space $(V_s,\oplus,\otimes)$. Furthermore, let $\mathbf{a}_{123}$ be a point in their gyroplane, which is off the gyrolines $\mathbf{a}_1\mathbf{a}_2, \mathbf{a}_2\mathbf{a}_3$, and $\mathbf{a}_3\mathbf{a}_1$. If $\mathbf{a}_1\mathbf{a}_{123}$ meets $\mathbf{a}_2\mathbf{a}_3$ at $\mathbf{a}_{23}$, etc., then*

$$\frac{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}}\|\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}\|}{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}}\|\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}\|}\frac{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{23}}\|\ominus\mathbf{a}_2\oplus\mathbf{a}_{23}\|}{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{23}}\|\ominus\mathbf{a}_3\oplus\mathbf{a}_{23}\|}\frac{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{13}}\|\ominus\mathbf{a}_3\oplus\mathbf{a}_{13}\|}{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{13}}\|\ominus\mathbf{a}_1\oplus\mathbf{a}_{13}\|} = 1,$$

(here $\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1-\frac{\|\mathbf{v}\|^2}{s^2}}}$ is the gamma factor).

(see [6, p. 461])

Theorem 1.2. (The Hyperbolic Theorem of Menelaus in Einstein Gyrovector Space). *Let* $\mathbf{a}_1, \mathbf{a}_2$*, and* $\mathbf{a}_3$ *be three non-gyrocollinear points in an Einstein gyrovector space* $(V_s, \oplus, \otimes)$*. If a gyroline meets the sides of gyrotriangle* $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$ *at points* $\mathbf{a}_{12}, \mathbf{a}_{13}, \mathbf{a}_{23},$ *then*

$$\frac{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}}\,\|\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}\|}{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}}\,\|\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}\|}\frac{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{23}}\,\|\ominus\mathbf{a}_2\oplus\mathbf{a}_{23}\|}{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{23}}\,\|\ominus\mathbf{a}_3\oplus\mathbf{a}_{23}\|}\frac{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{13}}\,\|\ominus\mathbf{a}_3\oplus\mathbf{a}_{13}\|}{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{13}}\,\|\ominus\mathbf{a}_1\oplus\mathbf{a}_{13}\|}=1.$$

(see [6, p. 463])

Theorem 1.3. (The Gyrotriangle Bisector Theorem). *Let* ABC *be a gyrotriangle in an Einstein gyrovector space* $(V_s, \oplus, \otimes)$*, and let* P *be a point lying on side* BC *of the gyrotriangle such that* AP *is a bisector of gyroangle* $\measuredangle BAC$*. Then,*

$$\frac{\gamma_{|BP|}\,|BP|}{\gamma_{|PC|}\,|PC|}=\frac{\gamma_{|AB|}\,|AB|}{\gamma_{|AC|}\,|AC|}.$$

(see [7, p. 150])

For further details we refer to the recent book of A.Ungar [6].

Definition 1.4. *The symmetric of the median with respect to the internal bisector issued from the same vertex is called symmedian.*

Theorem 1.5. *If the gyroline* AP *is a symmedian of a gyrotriangle* ABC, *and the point* P *is on the gyroside* BC*, then*

$$\frac{\gamma_{|CP|}\,|CP|}{\gamma_{|BP|}\,|BP|}=\left(\frac{\gamma_{|CA|}\,|CA|}{\gamma_{|BA|}\,|BA|}\right)^2.$$

(See [3])

Definition 1.6. *We call antibisector of a triangle, the izotomic of a internal bisector of a triangle interior angle.*

2. Main results

In this section, we present a proof of Pappus's harmonic theorem in the Einstein relativistic velocity model of hyperbolic geometry.

Theorem 2.1. (Pappus's harmonic theorem for hyperbolic gyrotriangle). *If* $A'B'C'$ *is the cevian gyrotriangle of gyropoint* M *with respect to the gyrotriangle* ABC *such that the gyrolines* $B'C'$ *and* BC *meet at* A''*, then*

$$\frac{\gamma_{|A'B|}|A'B|}{\gamma_{|A'C|}|A'C|}=\frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|}.$$

Proof. If we use Theorem 1.1 in the gyrotriangle ABC (see Figure 1), we have

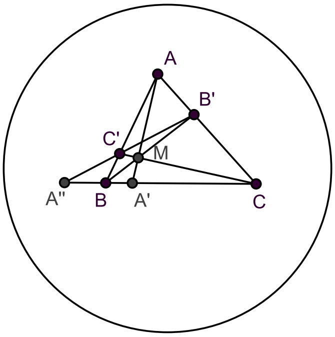

Figure 1

$$\frac{\gamma_{|A'B|}|A'B|}{\gamma_{|A'C|}|A'C|} \cdot \frac{\gamma_{|B'C|}|B'C|}{\gamma_{|B'A|}|B'A|} \cdot \frac{\gamma_{|C'A|}|C'A|}{\gamma_{|C'B|}|C'B|} = 1. \tag{2.1}$$

If we use Theorem 1.2 in the gyrotriangle ABC, cut by the gyroline $A'A''$, we get

$$\frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|} \cdot \frac{\gamma_{|B'C|}|B'C|}{\gamma_{|B'A|}|B'A|} \cdot \frac{\gamma_{|C'A|}|C'A|}{\gamma_{|C'B|}|C'B|} = 1. \tag{2.2}$$

From the relations (2.1) and (2.2) we have $\frac{\gamma_{|A'B|}|A'B|}{\gamma_{|A'C|}|A'C|} = \frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|}$. □

Corollary 2.2. *If $A'B'C'$ is the cevian gyrotriangle of gyropoint M with respect to the gyrotriangle ABC such that the gyrolines $B'C'$ and BC meet at A'', and AA' is a bisector of gyroangle $\measuredangle BAC$, then*

$$\frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|} = \frac{\gamma_{|AB|}|AB|}{\gamma_{|AC|}|AC|}.$$

Proof. If we use Theorem 1.3 in the triangle ABC, we get

$$\frac{\gamma_{|A'B|}|A'B|}{\gamma_{|A'C|}|A'C|} = \frac{\gamma_{|AB|}|AB|}{\gamma_{|AC|}|AC|}. \tag{2.3}$$

If we use Theorem 2.1 in the triangle ABC, we get

$$\frac{\gamma_{|A'B|}|A'B|}{\gamma_{|A'C|}|A'C|} = \frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|}. \tag{2.4}$$

From the relations (2.3) and (2.4) we have $\frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|} = \frac{\gamma_{|AB|}|AB|}{\gamma_{|AC|}|AC|}$. □

Corollary 2.3. *If $A'B'C'$ is the cevian gyrotriangle of gyropoint M with respect to the gyrotriangle ABC such that the gyrolines $B'C'$ and BC meet at A'', and*

AA' is a bisector of gyroangle $\measuredangle BAC$, and AA_1 is a antibisector of gyroangle $\measuredangle BAC$, then

$$\frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|} = \left(\frac{\gamma_{|A_1B|}|A_1B|}{\gamma_{|A_1C|}|A_1C|}\right)^{-1}.$$

Proof. Because the gyroline AA_1 is a isotomic line of the bisector AA', then

$$\frac{\gamma_{|A_1B|}|A_1B|}{\gamma_{|A_1C|}|A_1C|} = \frac{\gamma_{|A'C|}|A'C|}{\gamma_{|A'B|}|A'B|} = \frac{\gamma_{|AC|}|AC|}{\gamma_{|AB|}|AB|}. \tag{2.5}$$

If we use Corollary 2.2, we have

$$\frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|} = \frac{\gamma_{|AB|}|AB|}{\gamma_{|AC|}|AC|}. \tag{2.6}$$

From the relations (2.5) and (2.6), we have

$$\frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|} = \left(\frac{\gamma_{|A_1B|}|A_1B|}{\gamma_{|A_1C|}|A_1C|}\right)^{-1}. \tag{2.7}$$

□

Corollary 2.4. *If $A'B'C'$ is the cevian gyrotriangle of gyropoint M with respect to the gyrotriangle ABC such that the gyrolines $B'C'$ and BC meet at A'', and AA' is a symmedian of gyroangle $\measuredangle BAC$, and the point A' is on the gyroside BC, then*

$$\frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|} = \left(\frac{\gamma_{|AB|}|AB|}{\gamma_{|AC|}|AC|}\right)^{2}.$$

Proof. If we use Theorem 1.5, we have

$$\frac{\gamma_{|A'B|}|A'B|}{\gamma_{|A'C|}|A'C|} = \left(\frac{\gamma_{|AB|}|AB|}{\gamma_{|AC|}|AC|}\right)^{2}. \tag{2.8}$$

If we use Theorem 2.1, we have

$$\frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|} = \frac{\gamma_{|A'B|}|A'B|}{\gamma_{|A'C|}|A'C|}. \tag{2.9}$$

From the relations (2.8) and (2.9), we get $\frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|} = \left(\frac{\gamma_{|AB|}|AB|}{\gamma_{|AC|}|AC|}\right)^{2}$. □

Theorem 2.5. *If $A'B'C'$ is the cevian gyrotriangle of gyropoint M with respect to the gyrotriangle ABC such that the gyrolines $B'C'$ and BC meet at A'', and AA' is a bisector of gyroangle $\measuredangle BAC$, the gyrolines $A'C'$ and BB' meet at D, $A'B'$ and CC' meet at E, AD and BC meet at D', and AE and BC meet in E', then*

$$\frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|} = \frac{\gamma_{|D'B|}|D'B|}{\gamma_{|D'A'|}|D'A'|} \cdot \frac{\gamma_{|E'A'|}|E'A'|}{\gamma_{|E'C|}|E'C|}.$$

Proof. If we use Theorem 1.1 in the gyrotriangle ABA' (see Figure 2),

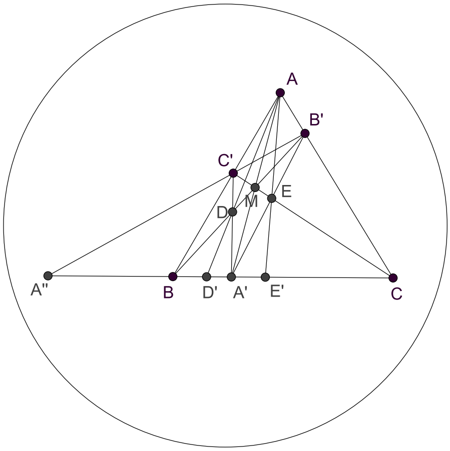

Figure 2

we have

$$\frac{\gamma_{|D'B|}|D'B|}{\gamma_{|D'A'|}|D'A'|} \cdot \frac{\gamma_{|C'A|}|C'A|}{\gamma_{|C'B|}|C'B|} \cdot \frac{\gamma_{|MA'|}|MA'|}{\gamma_{|MA|}|MA|} = 1. \tag{2.10}$$

If we use Theorem 1.2 in the gyrotriangle ABA', cut by the gyroline CC', we get

$$\frac{\gamma_{|CB|}|CB|}{\gamma_{|CA'|}|CA'|} \cdot \frac{\gamma_{|C'A|}|C'A|}{\gamma_{|C'B|}|C'B|} \cdot \frac{\gamma_{|MA'|}|MA'|}{\gamma_{|MA|}|MA|} = 1. \tag{2.11}$$

From the relations (2.10) and (2.11), we have

$$\frac{\gamma_{|D'B|}|D'B|}{\gamma_{|D'A'|}|D'A'|} = \frac{\gamma_{|CB|}|CB|}{\gamma_{|CA'|}|CA'|}. \tag{2.12}$$

Similarly, we obtain that

$$\frac{\gamma_{|E'C|}|E'C|}{\gamma_{|E'A'|}|E'A'|} = \frac{\gamma_{|BC|}|BC|}{\gamma_{|BA'|}|BA'|}. \tag{2.13}$$

If ratios the equations (2.12) and (2.13) among themselves, respectively, then

$$\frac{\gamma_{|D'B|}|D'B|}{\gamma_{|D'A'|}|D'A'|} \cdot \frac{\gamma_{|E'A'|}|E'A'|}{\gamma_{|E'C|}|E'C|} = \frac{\gamma_{|BA'|}|BA'|}{\gamma_{|CA'|}|CA'|}. \tag{2.14}$$

If we use Theorem 1.3 and the Corollary 2.2 in the triangle ABC, we get

$$\frac{\gamma_{|A'B|}|A'B|}{\gamma_{|A'C|}|A'C|} = \frac{\gamma_{|AB|}|AB|}{\gamma_{|AC|}|AC|} = \frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|}. \tag{2.15}$$

From the relations (2.14) and (2.15), we get

$$\frac{\gamma_{|D'B|}|D'B|}{\gamma_{|D'A'|}|D'A'|} \cdot \frac{\gamma_{|E'A'|}|E'A'|}{\gamma_{|E'C|}|E'C|} = \frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|}.$$

□

References

[1] Barbu, C., *Teoreme fundamentale din geometria triunghiului*, Ed. Unique, Bacău, 2008.

[2] Barbu, C., *Smarandache's cevian triangle theorem in the Einstein relativistic velocity model of hyperbolic geometry*, Progress in Physics, 2010.

[3] Barbu, C., *Steiner's Isogonals Theorem in the Einstein Relativistic Model of Hyperbolic Geometry*, Annales Mathematicae et Informaticae (to appear).

[4] Coxeter, H., Greitzer, S., *Geometry revisited*, Washington, DC, Math. Assoc. Amer., 1967, 67-68.

[5] Smarandache, F., *Eight Solved and Eight Open Problems in Elementary Geometry*, arXiv: 1003.2153.

[6] Ungar, A. A., *Analytic hyperbolic geometry and Albert Einstein's Special Theory of Relativity*, World Scientific Publishing Co., Hackensack (NJ), 2008.

[7] Ungar, A. A., *A Gyrovector Space Approach to Hyperbolic Geometry*, Morgan & Claypool Publishers, 2009.

Laurian-Ioan Pişcoran
North University of Baia Mare
Department of Mathematics and Computer Science
Victoriei 76, 430122 Baia Mare, Romania
e-mail: plaurian@yahoo.com

Cătălin Barbu
"Vasile Alecsandri" College
Bacău, str. Vasile Alecsandri nr. 37
600011 Bacău, Romania
e-mail: kafka_mate@yahoo.com