INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 1 (2012), No. 1, 10 - 19

TWO NEW PROOFS OF GOORMAGHTIGH'S THEOREM

ION PĂTRAŞCU and CĂTĂLIN I. BARBU

Abstract. In this note we present two new demonstrations of the theorem of a Belgian mathematician René Goormaghtigh.

1. Introduction

In order to state our main results we need recall some important theorems that we need in proving the Goormaghtigh's theorem. Consider a triangle ABC is neither isosceles rectangular nor with circumcenter O. We present below an interesting proposition given by Goormaghtigh.

Theorem 1.1. (*Goormaghtigh* [7, *pp.* 281 − 283]). *Let A', B', C' be points on OA, OB, OC so that*

$$\frac{OA'}{OA} = \frac{OB'}{OB} = \frac{OC'}{OC} = k,$$

$k \in \mathbb{R}_+^*$, then the intersections of the perpendiculars to OA at A', OB at B', and OC at C' with the respective sidelines BC, CA, AB are collinear.

R. Musselman and R. Goormaghtigh are given in [7] a proof of this theorem using complex numbers. A synthetic demonstration is also given by K. Nguyen meet in [9].

Theorem 1.2. (*Kariya* [3, *p.* 109]). *Let C_a, C_b, C_c the points of tangency of the incircle with the sides BC, CA, AB of triangle ABC and I center of the incircle. On the lines IC_a, IC_b, IC_c the points A', B', C' are considered in the same direction so that $IA' = IB' = IC'$. Then the lines AA', BB', and CC' are concurrent.*

Theorem 1.3. (*Desargues* [5, *p.* 133]). *Two triangles are in axial perspective if and only if they are in central perspective.*

Keywords and phrases: Goormaghtigh's theorem; Miquel's point; complete quadrilateral
(2010) Mathematics Subject Classification: 26D05; 26D15; 51N35

Theorem 1.4. (*Miquel* [6, *pp.* 233 − 234]). *The centers of the circles of the four triangles of a complete quadrilateral are on a circle. (Miquel's Circle).*

Theorem 1.5. (*Steiner* [6, *p.* 235]). *Miquel's point of the circles determined by the four triangles of a complete quadrilateral is situated on Miquel's circle.*

Theorem 1.6. (*Sondat* [11, *p.* 10]). *If two triangles ABC and $A'B'C'$ are perspective and orthologic, then the center of perspective P and the orthologic centers Q and Q' are on the same line perpendicular to the axis of perspectivity d.*

Theorem 1.7. (*Thébault* [12, *pp.* 22−24]). *If two triangles ABC and $A'B'C'$ are perspective and orthologic, with the center of perspective P and the orthologic centers Q and Q', then the conics $ABCPQ$ and $A'B'C'PQ'$ are equilateral hyperbolas.*

Theorem 1.8. (*Brianchon - Poncelet* [4, *pp.* 205 − 220]). *The centers of all equilateral hyperbolas passing through the vertices of a triangle ABC lie on the Euler circle of the triangle.*

2. Main results

In this section we present two new demonstrations of the theorem of a Belgian mathematician René Goormaghtigh and some consequences deriving from this theorem.

Solution 1. We noted with A'' the point of intersection of perpendiculars taken at B' and C' on the OB, OC respectively. Similarly we define the points B'' and C'' (see Figure 1).

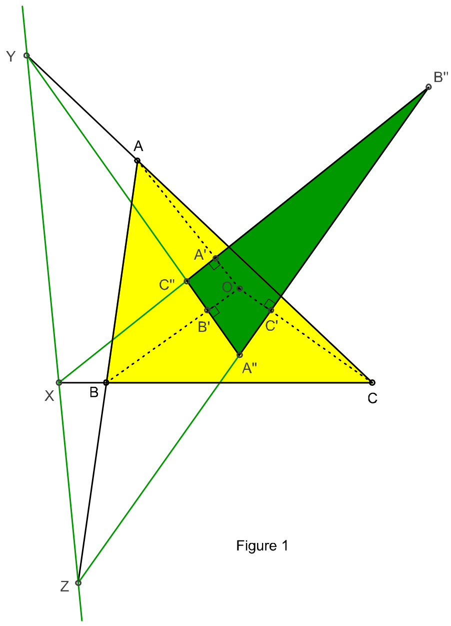

Figure 1

Since $OA = OB = OC$, from the relation

$$\frac{OA'}{OA} = \frac{OB'}{OB} = \frac{OC'}{OC} = k,$$

we get $OA' = OB' = OC'$. Because the lines OA', OB', OC' are perpendicular on $B''C'', C''A''$, and $A''B''$ respectively, then the point O is the incenter of the triangle $A''B''C''$. Applying theorem 2 in the triangle $A''B''C''$ for the points A, B, C, it results that the lines AA'', BB'' and CC'' are concurrent (at one of Kariya's points which corresponds to $A''B''C''$ triangle), then triangles ABC and $A''B''C''$ are homological. Thus, according to theorem 3, that the points of intersection of lines AB and $A''B''$, BC and $B''C''$, and CA and $C''A''$ are collinear.

Denote by X the intersection of the lines BC and $B''C''$. Similarly we define the points Y and Z. □

Solution 2. Without restricting the generality suppose that $\angle BCA > \angle ABC$. Let us designate by R the radius of the circle triangle ABC, by A_1 intersection of the tangent in A to circumcircle of the triangle ABC with the line CB, by T and X' the projections of the points B and X on this tangent, by M and M' the projections of points A' and O, respectively, with the line BT, and by A_1' the intersection of BC and OM' (see Figure 2).

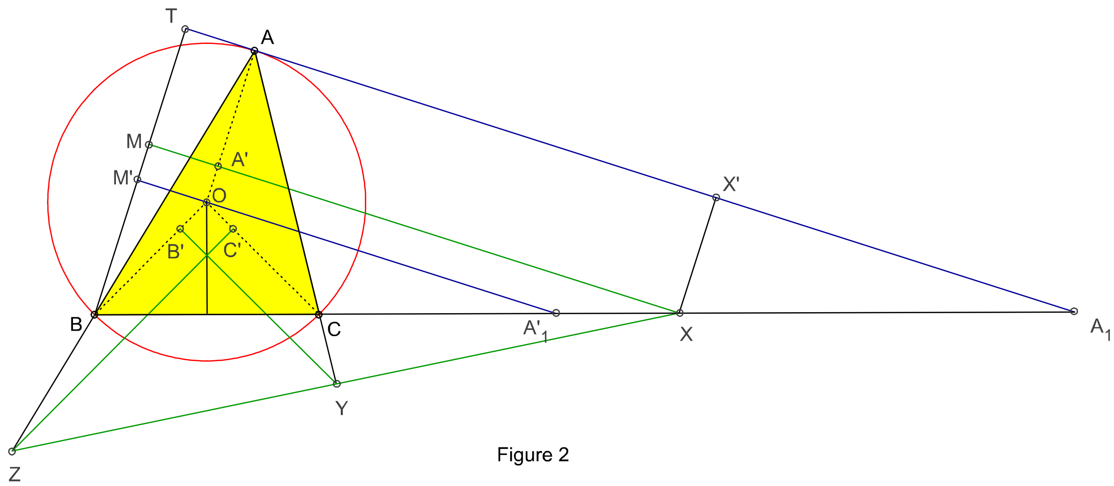

Figure 2

We have: $\angle CAA_1 = \angle ABC$, $\angle ACA_1 = \angle BAC + \angle ABC$ and

$$\angle AA_1B = 180^\circ - \angle BAC - 2 \cdot \angle ABC$$

$$= \angle BCA - \angle ABC, \angle COA_1' = 2 \cdot \angle ABC - 90^\circ.$$

Applying the law of sines in the triangle OCA_1, we have

$$\frac{A_1'C}{\sin(2B - 90^\circ)} = \frac{OC}{\sin(C - B)},$$

so

$$A_1'C = \frac{-R\cos 2B}{\sin(C - B)}.$$

Because $XX' = AA' = OA - OA' = R(1-k)$, then

$$XA_1 = \frac{XX'}{\sin(C-B)} = \frac{R(1-k)}{\sin(C-B)} \tag{1}$$

From $\frac{OA'}{OA} = \frac{A_1'X}{A_1'A_1} = k$, we get

$$\frac{A_1'X}{XA_1} = \frac{k}{1-k} \tag{2}$$

From relations (1) and (2) we get

$$A_1'X = \frac{k}{1-k} \cdot \frac{R(1-k)}{\sin(C-B)} = \frac{kR}{\sin(C-B)} \tag{3}$$

Since,

$$XC = XA_1' + A_1' = \frac{R(k - \cos 2B)}{\sin(C-B)} \tag{4}$$

Because $\angle M'OB = 2 \cdot \angle ACB - 90°$, $BM' = BO \cdot \sin(2C - 90°) = -R\cos 2C$, $MM' = OA' = kR$, then $BM = BM' + MM' = R(k - \cos 2C)$. Since

$$XB = \frac{BP}{\sin(C-B)} = \frac{R(k-\cos 2C)}{\sin(C-B)} \tag{5}$$

From relations (4) and (5) we get

$$\frac{XB}{XC} = \frac{k - \cos 2C}{k - \cos 2B}.$$

Similarly it is shown that

$$\frac{YC}{YA} = \frac{k - \cos 2A}{k - \cos 2C}$$

and

$$\frac{ZA}{ZB} = \frac{k - \cos 2B}{k - \cos 2A}.$$

We obtain that

$$\frac{XB}{XC} \cdot \frac{YC}{YA} \cdot \frac{ZA}{ZB} = 1$$

and from the converse of Menelaus's theorem results that points X, Y, and Z are collinear. □

Theorem 2.1. *Let us consider C_1, C_2, C_3 and C the circumcircles of the triangles AYZ, BZX, CXY, and respectively ABC. The circles C_1, C_2, C_3, and C pass through a common point.*

The proof results from theorem 5.

Let P be the corresponding point of Miquel complete quadrilateral $ABXYCZ$ (see Figure 3).

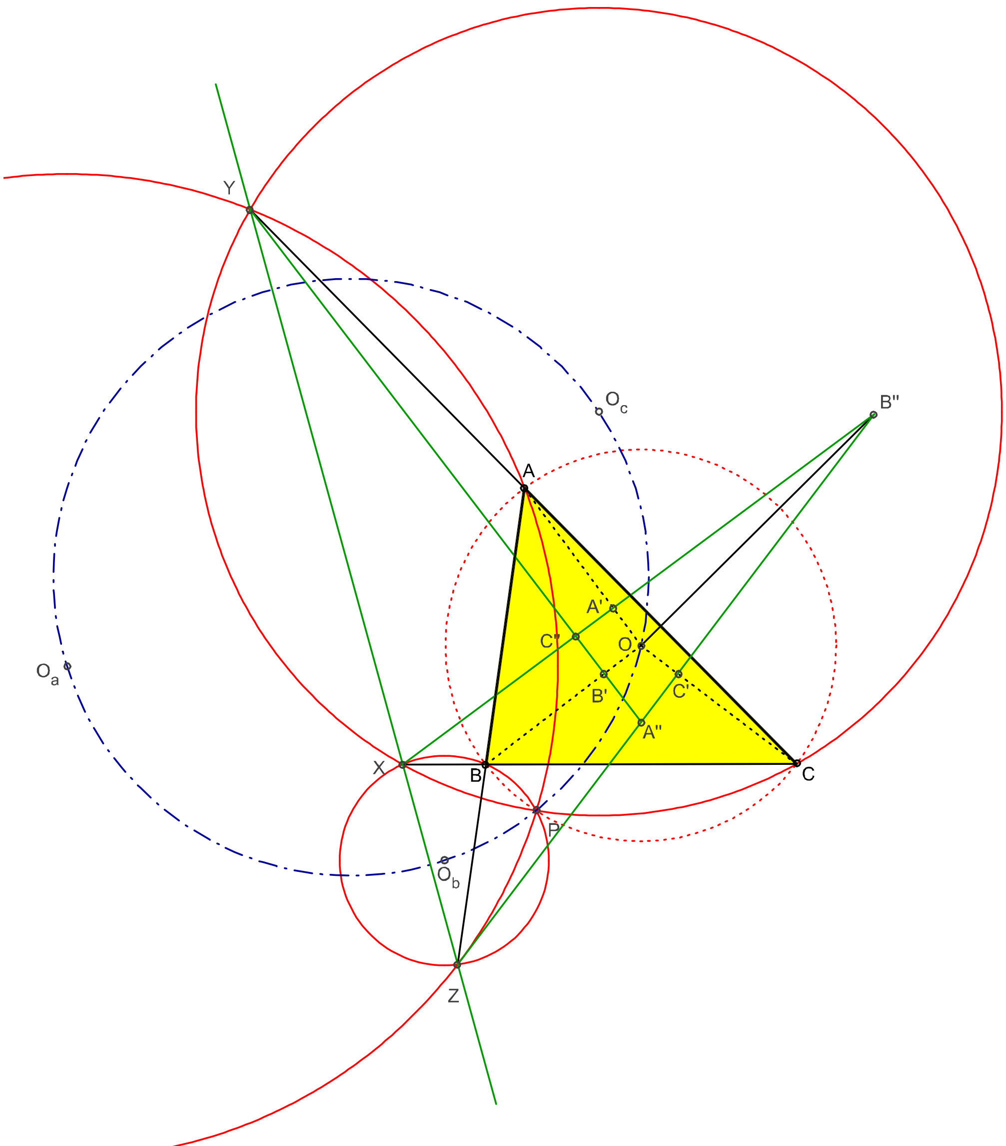

Figure 3

Theorem 2.2. *Centers of circles $C_1, C_2, C_3, \mathfrak{C}$ and point P are on the same circle $\aleph$.*

The proof results from theorems 4 and 5.

Theorem 2.3. *Let us consider C_1', C_2', and C_3' the circumcircles of the triangles $A''YZ$, $B''ZX$, and respectively $C''XY$. The circles C_1', C_2', and C_3' pass through a common point.*

The proof results from theorem 5.

Let us designate by $O_a, O_b, O_c, O_a', O_b', O_c'$ the circumcenters of the triangles AYZ, BZX, CXY, $A''YZ$, $B''ZX$, $C''XY$, respectively, by Q the point of Miquel of $A''C''XZB''Y$ complete quadrilateral (see Figure 4).

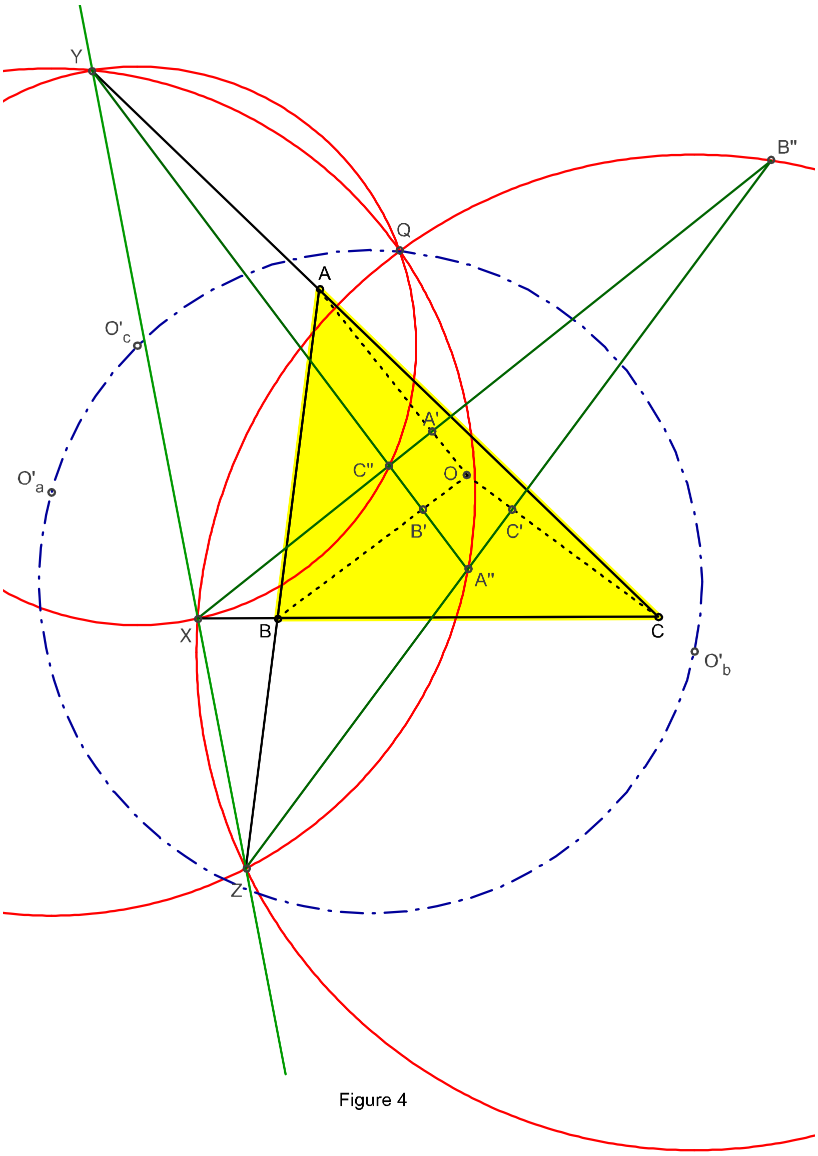

Figure 4

Open problems:

1) Point Q is on circle $\aleph$.

2) Point O is on Aubert's line of complete quadrilaterals $YABXCZ$ and $XZA''C''B''Y$.

Remark 2.1. *Goormaghtigh's theorem is true for $k < 0$, where $\overrightarrow{OA'} = k\overrightarrow{OA}$, $\overrightarrow{OB'} = k\overrightarrow{OB}$, $\overrightarrow{OC'} = k\overrightarrow{OC}$, the demonstration is similar.*

Remark 2.2. *Points A'', B'' and C'' are on the perpendicular bisectors of the sides of triangle ABC, therefore the triangles ABC and $A''B''C''$ are bilogical, O is a common center of orthology.*

Remark 2.3. *If $k = 0$ Goormaghtigh's theorem remains true as a special case of Bobillier's theorem* [10, *p.*119].

Remark 2.4. *For* $k = \frac{1}{2}$ *we obtain Ayme's theorem* [2].

Remark 2.5. *For* $k = 1$ *we obtain Lemoine's theorem and* XYZ *is Lemoine's line of the triangle* ABC [3, *p*.155].

Remark 2.6. *Theorem 4 is true for any transversal* XYZ *which cuts the sides of triangle* ABC*, the demonstration remains the same.*

Remark 2.7. *Because* $O_aO_a^{'}, O_bO_b^{'}, O_cO_c^{'}$ *are the perpendicular bisectors of the segments* YZ, ZX*, and* XY *respectively, then* $O_aO_a^{'} \| O_bO_b^{'} \| O_cO_c^{'}$.

Remark 2.8. *The triangles* ABC *and* $A''B''C''$ *are perspective,* XYZ *being the axis of perspective. Let* S *be the perspective center of triangles* ABC *and* $A''B''C''$*.*

Theorem 2.4. *The lines* OS *and* XYZ *are perpendicular.*

The proof results by Sondat's theorem (see Figure 5).

Theorem 2.5. *The conics* $ABCSO$ *and* $A'B'C'SO$ *are equilateral hyperbolas.*

Proof. Because the circumcenter O is the common center of orthology, by Theorem 1.7 we obtain the conclusion. □

Corollary 2.6. *The centers of the conics* $ABCSO$ *and* $A'B'C'SO$ *lie on the Euler circles of the triangles* ABC*, respectively* $A'B'C'$ *.*

The proof results from Theorems 1.8 and 2.5 (see Figure 5).

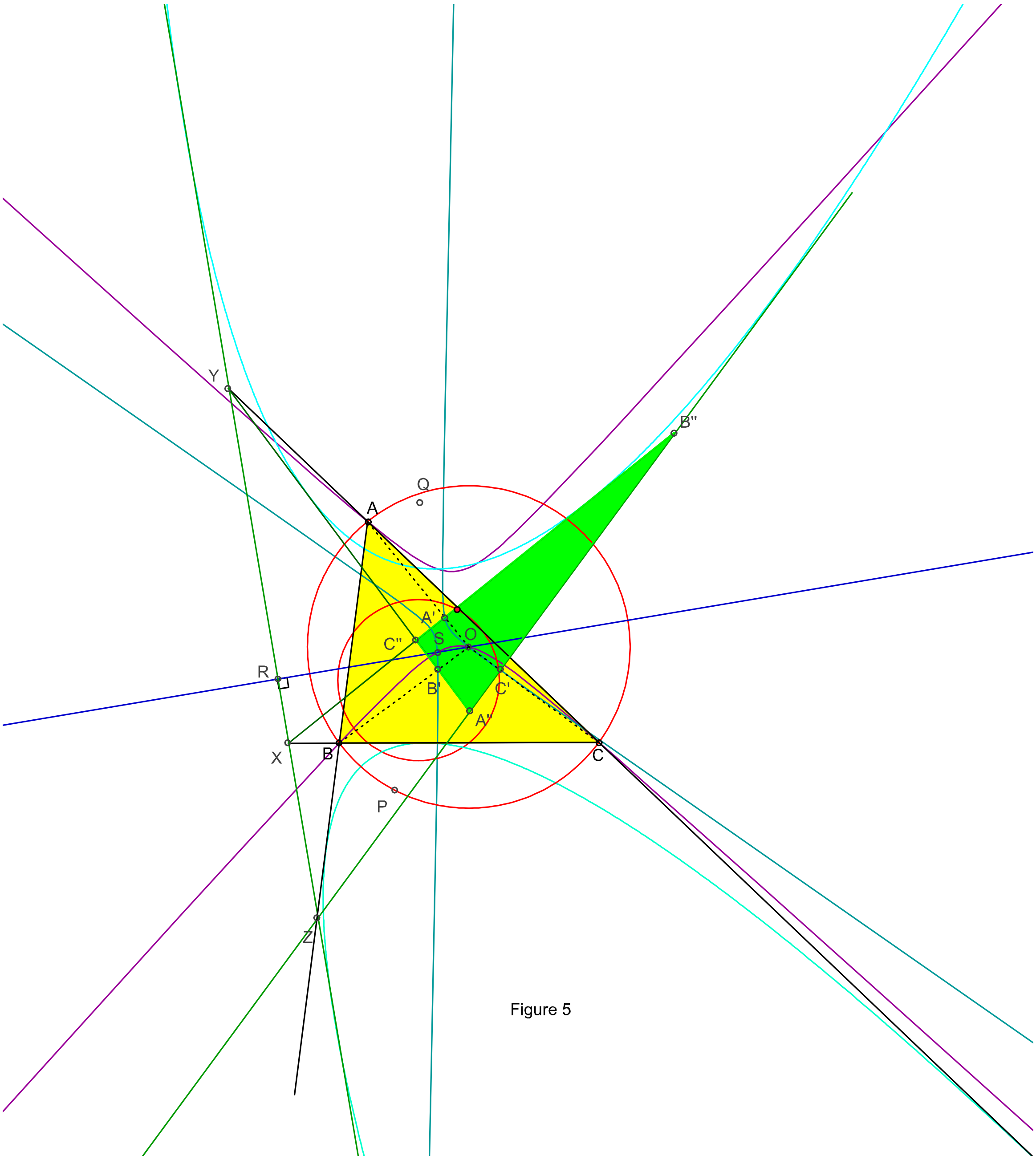

Figure 5

Corollary 2.7. *The points P ans Q are the focus of parabolas tangent to the sides of the complet quadrilaterals $ABXYCZ$ and $A''C''XZB''Y$, respectively.*

(see [1, p. 109], Figure 5).

3. Dynamic properties

In this section we examine the dependence of considered configuration on homothety coefficient k. Firstly formulate

Lemma 3.1. *Given two points A, B. The map f transforms the lines passing through A to the lines passing through B and conserve the cross-ratios of the lines. Then the locus of points $l \cap f(l)$ is a conic passing through A and B.*

Indeed if X, Y, Z are three points of the thought locus, then lines l and $f(l)$ intersect the conic $ABXYZ$ in the same point.

Lemma 3.1 has also a dual formulating: if f is a projective map between lines a and b then the envelop of lines $Af(A)$ is a conic touching a and b.

Using Lemma 3.1 we obtain that the envelop of lines XYZ from Theorem 1.1 is a parabola touching the sidelines of ABC, and the locus of perspectivity centers from Theorem 1.2 is the Feuerbach hyperbola.

Theorem 3.1. *Point P from Theorem 2.1 is fixed.*

Proof. Immediately follows from Lemma 3.1 and Corollary 2.7. □

Theorem 3.2. *The locus of points Q is a line passing through O.*

Proof. Using polar transformation with center O we obtain from Theorem 2.5 that two parabolas from Corollary 2.7 are homothetic. Thus all points Q are the foci of homothetic parabolas. □

Acknowledgement. The last section of the article was created by **Mr. Alexey Zaslavsky**, and the authors are grateful to him!

References

[1] Akopyan, A. V. and Zaslavsky, A. A., *Geometry of Conics*, American Mathematical Society, 2007, 109.
[2] Ayme, J. L. *Geometry*, *http://pagesperso-orange.fr/jl.ayme/*
[3] Barbu, C., *Fundamental Theorems of Triangle Geometry* (Romanian), Ed Unique, Bacău, 2008, 109.
[4] Brianchon, C. J. and Poncelet, J. V., *Recherches sur la détermination d'une hyperbole équiatère*, Gergonne's Annales de Math. **11(1820/21)**, 205-220.
[5] Court, N., *College Geometry*, Johnson Publishing Company, New York, 1925, 133.
[6] Mihalescu, C., *Remarkable Elements of Geometry* (Romanian), Ed. SSMR, Bucharest, 2007, 233-235.
[7] Musselman, J. R. and Goormaghtigh, R., *Advanced Problem* 3928, Amer. Math. Monthly, **46(1939)**, 601; solution, **48(1941)**, 281-283.
[8] Neuberg, V., *Mathesis*, 1922, 163.
[9] Nguyen, K. L., *A Synthetic Proof of Goormaghtigh's and Generalization of Musselman's Theorem*, Forum Geometricorum, **5(2005),** 17-20.
[10] Nicolescu, L. and Boscoff, V., *Practical Problems of Geometry* (Romanian), Technical Publishing House, Bucharest, 1990, 119.
[11] Sondat, P., *L'intermédiaire des mathématiciens*, 1894, 10 [question 38, solved by Sollerstinsky, 94].
[12] Thébault, V., *Perspective and Orthologic Triangles and Tetrahedrons*, Amer. Math. Monthly, **59(1952)**, 24-28.

Received: January 12, 2012.

FRAŢII BUZEŞTI COLLEGE
ION CANTACUZINO 15, S33, SC. 1, AP. 8, CRAIOVA, ROMANIA
E-mail address: `patrascu_ion@yahoo.com`

VASILE ALECSANDRI COLLEGE
IOSIF COCEA 12, SC. A, AP. 13, BACĂU, ROMANIA
E-mail address: `kafka_mate@yahoo.com`