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NOTE ON THE ADJOINT SPIEKER POINTS

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Abstract. In this note, we define and study the Speaker adjoint points of a triangle. The properties of some configurations involving these points are obtained by geometric methods and by using complex coordinates.

1. INTRODUCTION

Given a triangle ABC, denote by O the circumcenter, I the incenter, G the centroid, N the Nagel point, s the semiperimeter, R the circumradius, and r the inradius of ABC. Let M_a, M_b, M_c be the midpoints of the sides BC, CA, AB respectively. The Spieker point S_p of ABC is defined as the incenter of the median triangle $M_aM_bM_c$ of the triangle ABC. It is the center $X(10)$ in the Clark Kimberling's Encyclopedia of Triangle Centers and it has an important place in the modern triangle geometry. It is well-known that the line IN is called the Nagel line of the triangle ABC. The Spieker point S_p of ABC is situated on the Nagel line, and it is the midpoint of the segment $[IN]$. We shall introduce and study the points S_p^a, S_p^b, S_p^c, the adjoint points of the Spieker point S_p, and we show that they share similar properties as S_p (see Figure 1).

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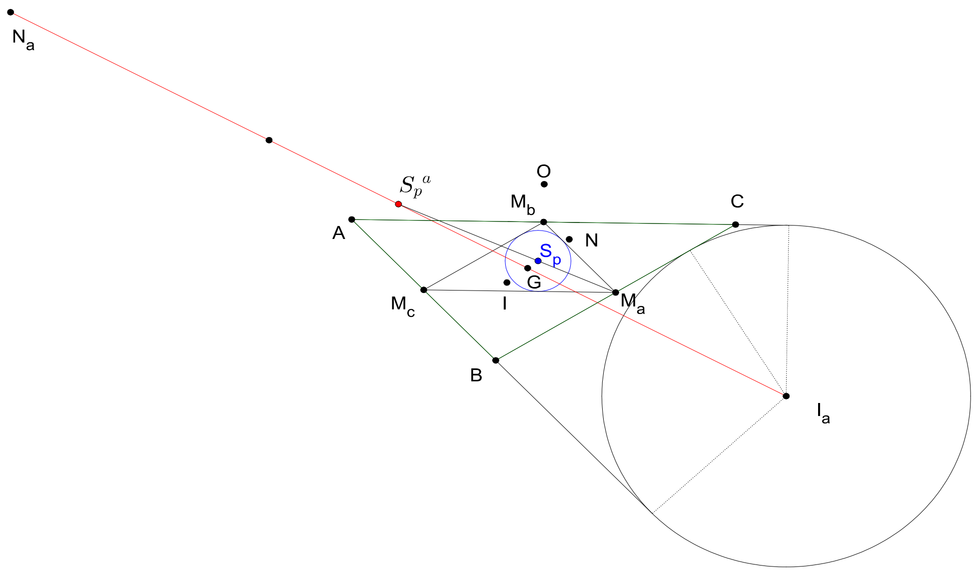

Figure 1

2. THE SPIEKER CONFIGURATION

In this section we consider an arbitrary triangle ABC with the circumcenter O, the incenter I, the excenters I_a, I_b, I_c, and N_a, N_b, N_c the adjoint points OF the Nagel point N. For the definition and some properties of the adjoint points N_a, N_b, N_c we refer to the paper of D. Andrica and K.L. Nguyen [2]. Let s, R, r, r_a, r_b, r_c be the semiperimeter, the circumradius, the inradius, and the exradii of triangle ABC, respectively. We know that points I_a, G, N_a are collinear and the relation $I_aN_a = 3I_aG$ holds. The similar properties hold for the triples of points N_b, G, I_b and N_c, G, I_c. These three lines are called the adjoint Nagel lines of the triangle ABC. The famous formulas for the distances OI_a and ON_a are given by

$$OI_a^2 = R^2 + 2Rr_a. \tag{1}$$

and

$$ON_a = R + 2r_a. \tag{2}$$

For a proof by using complex numbers we refer to the paper [2].

The **adjoint Spieker points** S_p^a, S_p^b, S_p^c of the Spieker point S_p of ABC are the excenters of the medial triangle $M_aM_bM_c$ of the triangle ABC. We called the triangle $S_p^aS_p^bS_p^c$ the **Spieker's triangle**.

Theorem 2.1. *The adjoint Spieker point S_p^a of ABC is the midpoint of the segment N_aI_a.*

Proof. Let S_1, S_2, S_3 be the midpoints of N_aI_a, N_bI_b, N_cI_c, respectively. First, we will prof that the triangles $S_1S_2S_3$.and $M_aM_bM_c$ are perspective at the Spieker point S_p. Because the angles $\widehat{AGI_a}$ and $\widehat{S_1GM_a}$ are equal and

$$\frac{AG}{GM_a} = \frac{I_aG}{GS_1} = 2,$$

it results that triangles AGI_a and M_aGS_1 are similar (see Figure 2).

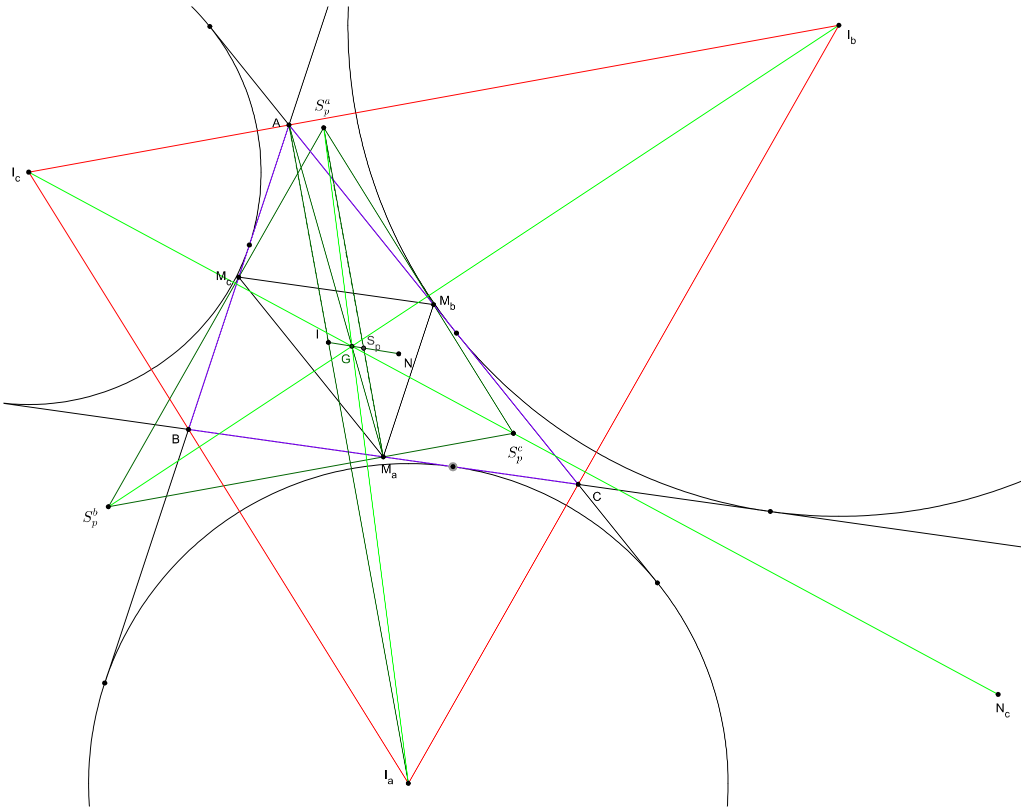

Figure 2

From this property we obtain $AI_a \| SM_a$. But $AB \| M_aM_b, AC \| M_aM_c$ and AI_a is the interior bisector of angle $\widehat{BAC}$, thus M_aS_1 is the bisector of $\widehat{M_bM_aM_c}$. We obtain that the Spieker point S_p of triangle ABC is situated on the line M_aS_1. Analogously, we get that S_p is on the lines M_bS_2 and

M_cS_3. Because

$$\frac{GI_a}{GS_1} = \frac{GI_b}{GS_2} = \frac{GI_c}{GS_3} = 2,$$

it follows that the triangle $S_1S_2S_3$ is homothetic to the external triangle $I_aI_bI_c$ by the homothety of center G and ratio -2. Since I is the orthocenter of triangle $I_aI_bI_c$ and S_p is imagine of I by this homotety, we get that S_p is the orthocenter of triangle $S_1S_2S_3$. Because $M_aM_bM_c$ is the orthic triangle of $S_1S_2S_3$, it follows that S_1, S_2, S_3 are the excenters of the median triangle $M_aM_bM_c$ of the triangle ABC. Thus we get the conclusion. □

Theorem 2.2. *The triangle $S_p^aS_p^bS_p^c$ is homothetic to $N_aN_bN_c$ by the homothety of center G and ratio* 3.

Proof.

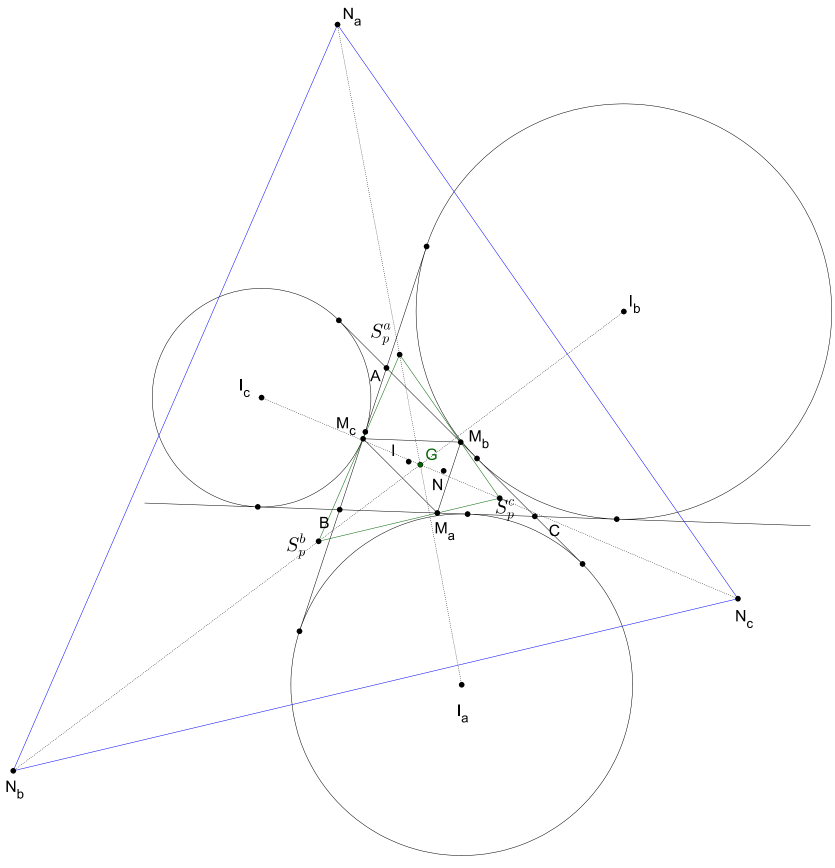

Figure 3

The lines $S_p^a N_a, S_p^b N_b$ and $S_p^c N_c$ are concurrent at G (see Figure 3), and we have the relations

$$\frac{GN_a}{GS_p^a} = \frac{GN_b}{GS_p^b} = \frac{GN_c}{GS_p^c} = 3.$$

□

Theorem 2.3. *The triangle $S_p^a S_p^b S_p^c$ is homothetic to the external triangle $I_a I_b I_c$ by the homothety of center G and ratio -2.*

Since I is the orthocenter of $I_a I_b I_c$, we obtain the following consequence.

Corollary 2.4. *The Spieker point S_p of triangle ABC is the orthocenter of triangle $S_p^a S_p^b S_p^c$.*

Corollary 2.5. *The median triangle $M_a M_b M_c$ of the triangle ABC is the orthic triangle of the Spieker triangle $S_p^a S_p^b S_p^c$.*

Corollary 2.6. *The triangles $S_p^a S_p^b S_p^c$ and $I_a I_b I_c$ are orthologic having I and S_p as centers of orthology.*

Corollary 2.7. *The Speaker triangle $S_p^a S_p^b S_p^c$ and the median triangle $M_a M_b M_c$ of ABC are in perspective at the Spieker point S_p.*

Corollary 2.8. *The points M_a, M_b, M_c lie on the sides of the Spieker triangle of ABC.*

3. COMPLEX COORDINATES APPROACH

In this section, we gave an alternative proof to Theorem 2.1 by using complex numbers.

Assume that the circumcenter O of triangle ABC is the origin of the complex plan and let z_A, z_B, z_C be the complex coordinates of A, B, C respectively. It is not difficult to calculate the complex coordinates of the points I_a and N_a (see [2]):

(3)
$$z_{I_a} = \frac{-az_A + bz_B + cz_C}{2(s-a)}$$

and

$$z_{N_a} = \frac{sz_A - (s-c)z_B - (s-b)z_C}{s-a}. \tag{4}$$

From the relations (3) and (4) we obtain the complex coordinates of the midpoint S of the segment I_aN_a:

$$z_S = \frac{z_{N_a} + z_{I_a}}{2} = \frac{(b+c)z_A + (c-a)z_B + (b-a)z_C}{4(s-a)}. \tag{5}$$

Using the complex coordinates of the midpoints M_a, M_b, M_c, it follows that

$$\begin{aligned} z_{S_p^a} &= \frac{-\frac{a}{2}z_{M_a} + \frac{b}{2}z_{M_b} + \frac{c}{2}z_{M_c}}{2\left(\frac{s}{2} - \frac{a}{2}\right)} \\ &= \frac{(b+c)z_A + (c-a)z_B + (b-a)z_C}{4(s-a)}. \end{aligned} \tag{6}$$

From the relations (5) and (6) we obtain the conclusion.

Remark 1. *Analogously, the adjoint Spieker points S_p^b and S_p^c of ABC are the midpoints of the segments I_bN_b and I_cN_c, respectively.*

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