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The Hyperbolic Version of Ceva's Theorem in the Poincaré Disc Model

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Abstract: In this note, we present the hyperbolic version of Ceva's theorem in the Poincaré disc model.

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1 Introduction

Hyperbolic Geometry appeared in the first half of the 19^{th} century as an attempt to understand Euclid's axiomatic basis of Geometry. It is also known as a type of non-Euclidean Geometry, being in many respects similar to Euclidean Geometry. Hyperbolic Geometry includes similar concepts as distance and angle. Both these geometries have many results in common but many are different.

There are known many models for Hyperbolic Geometry, such as: Poincaré disc model, Poincaré half-plane, Klein model, Einstein relativistic velocity model, etc. Here, in this study, we present a proof of Ceva's theorem in the Poincaré disc model of hyperbolic geometry. The Euclidean version of this well-known theorem states that if three lines from the vertices of a triangle $A_1A_2A_3$ are concurrent at M, and meet the opposite sides at P, Q, R respectively, then $\frac{A_1P}{PA_2} \cdot \frac{A_2R}{RA_3} \cdot \frac{A_3Q}{QA_1} = 1$ [7]. This result has a simple statement but it is of great interest. We just mention here few different proofs given by N.A.Court [3], D.Grindberg [5], R.Honsberg [6], A.Ungar [11].

We begin with the recall of some basic geometric notions and properties in the Poincaré disc. Let D denote the unit disc in the complex z - plane, i.e.$D = \{z \in \mathbb{C} : |z| < 1\}$ The most general Möbius transformation of D is

$$z \to e^{i\theta} \frac{z_0 + z}{1 + \overline{z_0} z} = e^{i\theta}(z_0 \oplus z),$$

which induces the Möbius addition $\oplus$ in D, allowing the Möbius transformation of the disc to be viewed as a Möbius left gyro-translation

$$z \to z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0} z}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $z, z_0 \in D$, and $\overline{z_0}$ is the complex conjugate of z_0. Let $Aut(D, \oplus)$ be the automorphism group of the grupoid $(D, \oplus)$.

If we define

$$gyr : D \times D \to Aut(D, \oplus), gyr[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\overline{b}}{1 + \overline{a}b},$$

then is true gyro-commutative law$a \oplus b = gyr[a, b](b \oplus a)$.

A gyro-vector space $(G, \oplus, \otimes)$ is a gyro-commutative gyro-group $(G, \oplus)$ that obeys the following axioms: *(1)*$gyr[\mathbf{u}, \mathbf{v}]\mathbf{a} \cdot gyr[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a} \cdot \mathbf{b}$ for all points $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in G$.

(2)G admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $\mathbf{a} \in G$:

(G1) $1 \otimes \mathbf{a} = \mathbf{a}$

(G2) $(r_1 + r_2) \otimes \mathbf{a} = r_1 \otimes \mathbf{a} \oplus r_2 \otimes \mathbf{a}$

(G3) $(r_1 r_2) \otimes \mathbf{a} = r_1 \otimes (r_2 \otimes \mathbf{a})$

(G4) $\frac{|r| \otimes \mathbf{a}}{\|r \otimes \mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$

(G5) $gyr[\mathbf{u}, \mathbf{v}](r \otimes \mathbf{a}) = r \otimes gyr[\mathbf{u}, \mathbf{v}]\mathbf{a}$

(G6) $gyr[r_1 \otimes \mathbf{v}, r_1 \otimes \mathbf{v}] = 1$

(3) Real vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of one-dimensional "vectors"

$$\|G\| = \{\pm \|\mathbf{a}\| : \mathbf{a} \in G\} \subset \mathbb{R}$$

with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in G$,

(G7) $\|r \otimes \mathbf{a}\| = |r| \otimes \|\mathbf{a}\|$

(G8) $\|\mathbf{a} \oplus \mathbf{b}\| \leq \|\mathbf{a}\| \oplus \|\mathbf{b}\|$

Theorem 1.1 ***(The law of gyrosines in Möbius gyrovector spaces).*** *Let ABC be a gyrotriangle in a Möbius gyrovector space $(V_s, \oplus, \otimes)$ with vertices $A, B, C \in V_s$, sides $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{V}_s$, and side gyrolengths $a, b, c \in (-s, s)$, $\mathbf{a} = \ominus B \oplus C$, $\mathbf{b} = \ominus C \oplus A$, $\mathbf{c} = \ominus A \oplus B$, $a = \|\mathbf{a}\|$, $b = \|\mathbf{b}\|$, $c = \|\mathbf{c}\|$, and with gyroangles α, β, and γ at the vertices A, B, and C. Then $\frac{a_\gamma}{\sin\alpha} = \frac{b_\gamma}{\sin\beta} = \frac{c_\gamma}{\sin\gamma}$, where $v_\gamma = \frac{v}{1-\frac{v^2}{s^2}}$ [10,p.267].*

Definition 1.2 *The hyperbolic distance function in D is defined by the equation*

$$d(a,b) = |a \ominus b| = \left|\frac{a-b}{1-\overline{a}b}\right|.$$

Here, $a \ominus b = a \oplus (-b)$, for $a, b \in D$.

For further details we refer to the recent book of A.Ungar [10].

Definition 1.3 *The symmetric of the median of a triangle with respect to the internal bisector issued from the same vertex is called symmedian.*

Theorem 1.4 (***The Gyrotriangle Bisector Theorem***). *Let ABC be a gyrotriangle in a Möbius gyrovector space $(V_s, \oplus, \otimes)$ with vertices $A, B, C \in V_s$, sides $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{V}_s$, and side gyrolengths $a, b, c \in (-s, s)$, $\mathbf{a} = \ominus B \oplus C$, $\mathbf{b} = \ominus C \oplus A$, $\mathbf{c} = \ominus A \oplus B$, $a = \|\mathbf{a}\|$, $b = \|\mathbf{b}\|$, $c = \|\mathbf{c}\|$, and let D be a point lying on side BC of the gyrotriangle such that AD is a bisector of gyroangle $\angle BAC$. Then*

$$\frac{(DB)_\gamma}{(DC)_\gamma} = \frac{(AB)_\gamma}{(AC)_\gamma},$$

where $v_\gamma = \frac{v}{1-\frac{v^2}{s^2}}$ [1].

2 Main results

In this section we prove the Ceva's theorem in the Poincaré disc model of hyperbolic geometry.

Theorem 2.1 ***(The Ceva's Theorem for Hyperbolic Gyrotriangle)*** *If M is a point not on any side of a gyrotriangle $A_1A_2A_3$ such that A_3M and A_1A_2 meet in P, A_2M and A_3A_1 in Q, and A_1M and A_2A_3 meet in R, then*

$$\frac{(A_1P)_\gamma}{(A_2P)_\gamma} \cdot \frac{(A_2R)_\gamma}{(A_3R)_\gamma} \cdot \frac{(A_3Q)_\gamma}{(A_1Q)_\gamma} = 1$$

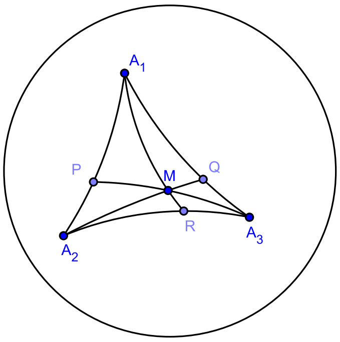

Figure 1

Proof. The law of gyrosines (See Theorem 1.1), gives for the gyrotriangles A_1MP and A_1MP (See Figure 1) respectively,

$$\frac{(A_1P)_\gamma}{\sin \widehat{A_1MP}} = \frac{(A_1M)_\gamma}{\sin \widehat{A_1PM}} \tag{1}$$

$$\frac{(A_2P)_\gamma}{\sin \widehat{A_2MP}} = \frac{(A_2M)_\gamma}{\sin \widehat{A_2PM}} \tag{2}$$

where $\sin \widehat{A_1PM} = \sin \widehat{A_2PM}$ since gyroangles $\widehat{A_1PM}$ and $\widehat{A_2PM}$ are suplementary. Hence, by (1) and (2), we have

$$\frac{(A_1P)_\gamma}{(A_2P)_\gamma} = \frac{(A_1M)_\gamma}{(A_2M)_\gamma} \cdot \frac{\sin \widehat{A_1MP}}{\sin \widehat{A_2MP}} = \frac{(A_1M)_\gamma}{(A_2M)_\gamma} \cdot \frac{\sin \widehat{A_1MA_3}}{\sin \widehat{A_2MA_3}} \tag{3}$$

Similary, applying the law of gyrosines to the pair of gyrotriangles A_2MR and A_3MR, we have

$$\frac{(A_2R)_\gamma}{(A_3R)_\gamma} = \frac{(A_2M)_\gamma}{(A_3M)_\gamma} \cdot \frac{\sin \widehat{A_2MA_1}}{\sin \widehat{A_3MA_1}} \tag{4}$$

and applying the law of gyrosines to the pair of gyrotriangles A_2MR and A_3MR, we have

$$\frac{(A_3Q)_\gamma}{(A_1Q)_\gamma} = \frac{(A_3M)_\gamma}{(A_1M)_\gamma} \cdot \frac{\sin \widehat{A_3MA_2}}{\sin \widehat{A_1MA_2}} \tag{5}$$

Now, from (3)-(5) we obtain

$$\frac{(A_1P)_\gamma}{(A_2P)_\gamma} \cdot \frac{(A_2R)_\gamma}{(A_3R)_\gamma} \cdot \frac{(A_3Q)_\gamma}{(A_1Q)_\gamma} =$$

$$\left(\frac{(A_1M)_\gamma}{(A_2M)_\gamma} \cdot \frac{\sin \widehat{A_1MA_3}}{\sin \widehat{A_2MA_3}}\right) \cdot \left(\frac{(A_2M)_\gamma}{(A_3M)_\gamma} \cdot \frac{\sin \widehat{A_2MA_1}}{\sin \widehat{A_3MA_1}}\right) \cdot \left(\frac{(A_3M)_\gamma}{(A_1M)_\gamma} \cdot \frac{\sin \widehat{A_3MA_2}}{\sin \widehat{A_1MA_2}}\right) = 1$$

■

Naturally, one may wonder whether the converse of the Ceva theorem exists. Indeed, a partially converse theorem does exist as we show in the following theorem.

Theorem 2.2 (***Converse of Ceva's Theorem for Hyperbolic Gyrotriangle***) *If P lies* *on the gyroline A_1A_2, R on A_2A_3, and Q on A_3A_1 such that*

$$\frac{(A_1P)_\gamma}{(A_2P)_\gamma}\cdot\frac{(A_2R)_\gamma}{(A_3R)_\gamma}\cdot\frac{(A_3Q)_\gamma}{(A_1Q)_\gamma}=1, \tag{*}$$

and two of the gyrolines A_1R, A_2Q and A_3P meet, then all three are concurrent.

Proof. If P lies between A_1 and A_2, then A_3P cuts the gyrosegment A_1R in M. Also, A_2M cuts gyroside A_3A_1 in Q'. Applying Ceva's theorem to the gyrotriangle $A_1A_2A_3$ and the point M, we get

$$\frac{(A_1P)_\gamma}{(A_2P)_\gamma}\cdot\frac{(A_2R)_\gamma}{(A_3R)_\gamma}\cdot\frac{(A_3Q')_\gamma}{(A_1Q')_\gamma}=1 \tag{6}$$

From $(*)$ and (6), we get $\frac{(A_3Q)_\gamma}{(A_1Q)_\gamma}=\frac{(A_3Q')_\gamma}{(A_1Q')_\gamma}$. This equation holds for $Q=Q'$. Indeed, if we take $x:=|\ominus A_3\oplus Q'|$ and $b:=|\ominus A_3\oplus A_1|$, then we get $b\ominus x=|\ominus Q'\oplus A_1|$. For $x\in(-1,1)$ define

$$f(x)=\frac{x}{1-x^2}:\frac{b\ominus x}{1-(b\ominus x)^2}.$$

Because $b\ominus x=\frac{b-x}{1-bx}$, then $f(x)=\frac{x(1-b^2)}{(b-x)(1-bx)}$. Since the following equality holds

$$f(x)-f(y)=\frac{b(1-b^2)(1-xy)}{(b-x)(1-bx)(b-y)(1-by)}(x-y),$$

we get $f(x)$ is an injective function and this implies $Q=Q'$. A similar argument applies if Q lies between A_1 and A_3.

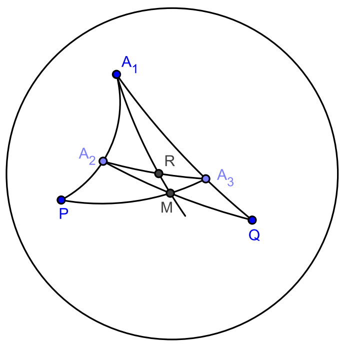

Figure 2

Now suppose that P is situated beyond A_2, and Q beyond A_3, then the gyrolines A_2Q and A_3P meet at M, which lies within the gyroangle $A_2A_1A_3$ (See Figure 2). Now A_1M cuts the gyrosegment A_2A_3 in the gyropoint R'. Consequently $R=R'$, so the gyrolines are concurrent. Next suppose that the gyropoint P is situated beyond A_2, and Q beyond A_1 (See Figure 3).

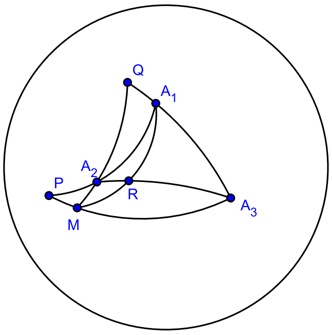

Figure 3

Then the gyroline A_2Q enters gyroangle PA_2A_3 at A_2, and so cuts A_3P at M. Since M is situated within the gyroangle $A_2A_1A_3$, A_1M cuts the gyrosegment A_2A_3 in the gyropoint R'. As a consequence $R = R'$, so the gyrolines are concurrent. The case where P is beyond A_1, and Q beyond A_3 is similar. There are with cases where both P and Q are situated beyond A_1 (See Figure 4).

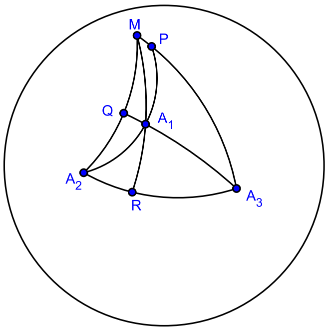

Figure 4

By using the hypotheses we suppose first that the gyrolines A_2Q and A_3P meet in the gyropoint M. Then M is situated within gyroangle QAP, so A_1M meets the gyrosegment A_2A_3 in the gyropoint R'. Consequently $R = R'$, so the gyrolines are concurrent. Suppose next that A_2Q and A_3P meet in the gyropoint M, then M and A_3 lie on opposite sides of A_1A_2, so A_3M meets A_1A_2 in the gyropoint P'. Consequently $P = P'$, so the gyrolines are concurrent. ■

Corollary 2.3 *The gyromedians of a gyrotriangle $A_1A_2A_3$ are concurrent.*

Proof. Let P, Q, R are the midpoints of the gyrosides A_2A_1, A_1A_3, and A_3A_1 respectively (See Figure 1). Because $(A_1P)_\gamma = (A_2P)_\gamma, (A_2R)_\gamma = (A_3R_1)_\gamma$, and $(A_3Q)_\gamma = (A_1Q)_\gamma$, then

$$\frac{(A_1P)_\gamma}{(A_2P)_\gamma} \cdot \frac{(A_2R)_\gamma}{(A_3R)_\gamma} \cdot \frac{(A_3Q)_\gamma}{(A_1Q)_\gamma} = 1.$$

The gyromedians all lie within the gyrotriangle, so any two must meet. Thus, by the Converse of Ceva's Theorem, the gyromedians A_1R, A_2Q and A_3P are concurrent. ■

Theorem 2.4 ***(The Hyperbolic Theorem of Steiner).*** *If the gyrolines A_1P and A_1Q are two isogonals of a vertex A_1 of a gyrotriangle $A_1A_2A_3$, and the gyropoints P and Q are* *on the gyroside A_2A_3, then*

$$\frac{(CQ)_\gamma}{(A_2Q)_\gamma}\cdot\frac{(CP)_\gamma}{(A_2P)_\gamma}=\left(\frac{(CA_1)_\gamma}{(A_2A_1)_\gamma}\right)^2.$$

Proof. We set $\angle A_2A_1Q=\angle PA_1A_3=\theta, \angle A_1QA_2=\epsilon_1, \angle A_1QA_3=\epsilon_2, \angle A_1PA_2=\lambda_1, \angle A_1PA_3=\lambda_2$ (See Figure 5).

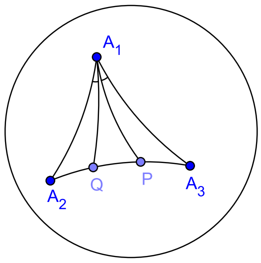

Figure 5

If we use the gyrosines theorem in the triangles $A_1A_2Q, A_1A_3Q, A_1A_3P, A_1A_2P$ respectively (See Theorem 1.1), then

$$\frac{\sin\theta}{(A_2Q)_\gamma}=\frac{\sin\epsilon_1}{(A_2A_1)_\gamma}, \tag{7}$$

$$\frac{\sin(A_1-\theta)}{(A_3Q)_\gamma}=\frac{\sin\epsilon_2}{(A_3A_1)_\gamma}, \tag{8}$$

$$\frac{\sin\theta}{(A_3P)_\gamma}=\frac{\sin\lambda_2}{(A_3A_1)_\gamma}, \tag{9}$$

$$\frac{\sin(A_1-\theta)}{(A_2P)_\gamma}=\frac{\sin\lambda_1}{(A_2A_1)_\gamma}. \tag{10}$$

If ratios the equations (7) and (8) among themselves, respectively, and because $\sin(\pi-\theta)=\sin\theta$, then

$$\frac{\sin\theta}{\sin(A_1-\theta)}\cdot\frac{(A_3Q)_\gamma}{(A_2Q)_\gamma}=\frac{(A_3A_1)_\gamma}{(A_2A_1)_\gamma}. \tag{11}$$

If ratios the equations (9) and (10) among themselves, respectively, then

$$\frac{\sin\theta}{\sin(A_1-\theta)}\cdot\frac{(A_2P)_\gamma}{(A_3P)_\gamma}=\frac{(A_2A_1)_\gamma}{(A_3A_1)_\gamma}. \tag{12}$$

If ratios the equations (11) and (12) among themselves, respectively, then

$$\frac{(A_3Q)_\gamma}{(A_2Q)_\gamma}\cdot\frac{(A_3P)_\gamma}{(A_2P)_\gamma}=\left(\frac{(A_3A_1)_\gamma}{(A_2A_1)_\gamma}\right)^2. \tag{13}$$

■

Corollary 2.5 *If the gyroline A_1P is a gyrosymmedian of a gyrotriangle $A_1A_2A_3$, and the point P is on the gyroside A_2A_3, then*

$$\frac{(A_3P)_\gamma}{(A_2P)_\gamma} = \left(\frac{(A_3A_1)_\gamma}{(A_2A_1)_\gamma}\right)^2. \tag{14}$$

Proof. Let A_1Q be the gyromedian in the gyrotriangle $A_1A_2A_3$ (See Figure 5). If we use theorem 2.3 for the isogonals A_1P and A_1Q, we obtain

$$\frac{(A_3Q)_\gamma}{(A_2Q)_\gamma} \cdot \frac{(A_3P)_\gamma}{(A_2P)_\gamma} = \left(\frac{(A_3A_1)_\gamma}{(A_2A_1)_\gamma}\right)^2.$$

Because $(A_3Q)_\gamma = (A_2Q)_\gamma$, the conclusion follows. ■

Corollary 2.6 *The gyrosymedians of a gyrotriangle are concurrent.*

Proof. Let $A_1A_2A_3$ be a gyrotriangle, and let the gyrosymedians be A_1R, A_2Q and A_3P. If we use the Corollary 2.2 we get

$$\frac{(A_3R)_\gamma}{(A_2R)_\gamma} = \left(\frac{(A_3A_1)_\gamma}{(A_2A_1)_\gamma}\right)^2, \tag{15}$$

and

$$\frac{(A_1Q)_\gamma}{(A_3Q)_\gamma} = \left(\frac{(A_1A_2)_\gamma}{(A_3A_2)_\gamma}\right)^2, \tag{16}$$

and

$$\frac{(A_2P)_\gamma}{(A_1P)_\gamma} = \left(\frac{(A_2A_3)_\gamma}{(A_1A_3)_\gamma}\right)^2. \tag{17}$$

From (15), (16), and (17) we obtain

$$\frac{(A_3R)_\gamma}{(A_2R)_\gamma} \cdot \frac{(A_1Q)_\gamma}{(A_3Q)_\gamma} \cdot \frac{(A_2P)_\gamma}{(A_1P)_\gamma} = 1,$$

and from Theorem 2.2 we get the conclusion. ■

Corollary 2.7 *The internal angle bisectors of a gyrotriangle $A_1A_2A_3$ are concurrent.*

Proof. Let $A_1A_2A_3$ be a gyrotriangle, and let the angle bisectors be A_1R, A_2Q and A_3P (See Figure 1). If we use the Theorem 1.2 we get

$$\frac{(A_3R)_\gamma}{(A_2R)_\gamma} = \frac{(A_3A_1)_\gamma}{(A_2A_1)_\gamma}, \tag{18}$$

and

$$\frac{(A_1Q)_\gamma}{(A_3Q)_\gamma} = \frac{(A_1A_2)_\gamma}{(A_3A_2)_\gamma}, \tag{19}$$

and

$$\frac{(A_2P)_\gamma}{(A_1P)_\gamma} = \frac{(A_2A_3)_\gamma}{(A_1A_3)_\gamma}. \tag{20}$$

From (18), (19), and (20) we obtain

$$\frac{(A_3R)_\gamma}{(A_2R)_\gamma} \cdot \frac{(A_1Q)_\gamma}{(A_3Q)_\gamma} \cdot \frac{(A_2P)_\gamma}{(A_1P)_\gamma} = 1.$$

The angle bisectors all lie within the gyrotriangle, so any two must meet. Thus, by the Converse of Ceva's Theorem, the internal angle bisectors A_1R, A_2Q and A_3P are concurrent. ∎

Many of the theorems of Euclidean geometry are relatively similar form in the Poincaré model of hyperbolic geometry, Ceva's theorem and Stewart's theorem are an examples in this respect. In the Euclidean limit of large s, $s \to \infty$, v_γ reduces to v, so the relations $(*)$ and (13) reduces to

$$\frac{A_1P}{A_2P} \cdot \frac{A_2R}{A_3R} \cdot \frac{A_3Q}{A_1Q} = 1,$$

and

$$\frac{A_3Q}{A_2Q} \cdot \frac{A_3P}{A_2P} = \left(\frac{A_3A_1}{A_2A_1}\right)^2$$

in euclidian geometry.

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