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THE HYPERBOLIC STEWART THEOREM IN THE EINSTEIN RELATIVISTIC VELOCITY MODEL OF HYPERBOLIC GEOMETRY

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Abstract. In this study, we present a proof of the hyperbolic Stewart theorem in the Einstein relativistic velocity model of hyperbolic geometry.

1. Introduction

Hyperbolic geometry appeared in the first half of the 19^{th} century as an attempt to understand Euclid's axiomatic basis of geometry. It is also known as a type of non-Euclidean geometry, being in many respects similar to euclidean geometry. Hyperbolic geometry includes similar concepts as distance and angle. Both these geometries have many results in common but many are different. Several useful models of hyperbolic geometry are studied in the literature as, for instance, the Poincaré disc and ball models, the Poincaré half-plane model, and the Beltrami-Klein disc and ball models etc. Following [5] and [8] and earlier discoveries, the Beltrami-Klein model is also known as the Einstein relativistic velocity model. Here, in this study, we present a proof of the hyperbolic Stewart theorem in the Einstein relativistic velocity model of hyperbolic geometry. The well-known Stewart theorem states that if a point D lies between the vertices A and C of the triangle ABC, then $AB^2 \cdot DC + BC^2 \cdot AD - BD^2 \cdot AC = AC \cdot DC \cdot AD$ ([1, p 152]). This result has a simple statement but it is of great interest. We just mention here few different proofs given by O. Demirel [2], W. Stothers [3], V. Boskoff [4].

Let D denote the complex unit disc in complex z - plane, i.e.

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

The most general Möbius transformation of D is

$$z \to e^{i\theta} \frac{z_0 + z}{1 + \overline{z_0} z} = e^{i\theta}(z_0 \oplus z),$$

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which induces the Möbius addition $\oplus$ in D, allowing the Möbius transformation of the disc to be viewed as a Möbius left gyrotranslation

$$z \to z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0} z}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $z, z_0 \in D$, and $\overline{z_0}$ is the complex conjugate of z_0. Let $Aut(D, \oplus)$ be the automorphism group of the grupoid $(D, \oplus)$. If we define

$$gyr : D \times D \to Aut(D, \oplus), gyr[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\overline{b}}{1 + \overline{a}b},$$

then the gyrocommutative law $a \oplus b = gyr[a, b](b \oplus a)$ is satisfied.

Definition 1.1. A groupoid $(G, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms. In G there is at least one element, 0, called a left identity, satisfying

$$(G1) \quad 0 \oplus a = a$$

for all $a \in G$. There is an element $0 \in G$ satisfying axiom $(G1)$ such that for each $a \in G$ there is an element $\ominus a \in G$, called a left inverse of a, satisfying

$$(G2) \quad \ominus a \oplus a = 0$$

Moreover, for any $a, b, c \in G$ there exists a unique element $gyr[a, b]c \in G$ such that the binary operation obeys the left gyroassociative law

$$(G3) \quad a \oplus (b \oplus c) = (a \oplus b) \oplus gyr[a, b]c$$

The map $gyr[a, b] : G \to G$ given by $c \mapsto gyr[a, b]c$ is an automorphism of the groupoid $(G, \oplus)$,

$$(G4) \quad gyr[a, b] \in Aut(G, \oplus)$$

and the automorphism $gyr[a, b]$ of G is called the gyroautomorphism of G generated by $a, b \in G$. The operator $gyr : G \times G \to Aut(G, \oplus)$ is called the gyrator of G. Finally, the gyroautomorphism $gyr[a, b]$ generated by any $a, b \in G$ possesses the left loop property

$$(G5) \quad gyr[a, b] = gyr[a \oplus b, b]$$

(see [5, p.17])

A gyrovector space $(G, \oplus, \otimes)$ is a gyrocommutative gyrogroup $(G, \oplus)$ that obeys the following axioms:

(1) $gyr[\mathbf{u}, \mathbf{v}]\mathbf{a} \cdot gyr[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a} \cdot \mathbf{b}$ for all points $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in G$.

(2) G admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $\mathbf{a} \in G$:

(G1) $1 \otimes \mathbf{a} = \mathbf{a}$

(G2) $(r_1 + r_2) \otimes \mathbf{a} = r_1 \otimes \mathbf{a} \oplus r_2 \otimes \mathbf{a}$

(G3) $(r_1 r_2) \otimes \mathbf{a} = r_1 \otimes (r_2 \otimes \mathbf{a})$

(G4) $\frac{|r| \otimes \mathbf{a}}{\|r \otimes \mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$

(G5) $gyr[\mathbf{u},\mathbf{v}](r\otimes\mathbf{a}) = r\otimes gyr[\mathbf{u},\mathbf{v}]\mathbf{a}$

(G6) $gyr[r_1\otimes\mathbf{v}, r_1\otimes\mathbf{v}] = 1$

(3) Real vector space structure $(\|G\|,\oplus,\otimes)$ for the set $\|G\|$ of onedimensional "vectors"

$$\|G\| = \{\pm\|\mathbf{a}\| : \mathbf{a}\in G\}\subset\mathbb{R}$$

with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r\in\mathbb{R}$ and $\mathbf{a},\mathbf{b}\in G$,

(G7) $\|r\otimes\mathbf{a}\| = |r|\otimes\|\mathbf{a}\|$

(G8) $\|\mathbf{a}\oplus\mathbf{b}\|\le\|\mathbf{a}\|\oplus\|\mathbf{b}\|$

Definition 1.2. Let $G=(G,\oplus,\otimes)$ be a gyrovector space. Its gyrometric is given by the gyrodistance function $d(\mathbf{a},\mathbf{b}): G\times G\to[0,\infty)$,

$$d(\mathbf{a},\mathbf{b}) = \|\ominus\mathbf{a}\oplus\mathbf{b}\| = \|\mathbf{b}\ominus\mathbf{a}\|$$

where $d(\mathbf{a},\mathbf{b})$ is the gyrodistance of $\mathbf{a}$ to $\mathbf{b}$.

(see [5, p.157])

Definition 1.3. A gyrotriangle ABC in a gyrovector space $(G,\oplus,\otimes)$ is a gyrovector space object formed by the three points $A,B,C\in G$, called the vertices of the gyrotriangle, and the gyrosegments AB, AC and BC, called the sides of the gyrotriangle. These are, respectively, the sides opposite to the vertices C,B and A. The gyrotriangle sides generate the three gyrotriangle gyroangles, α,β, and γ, $0<\alpha,\beta,\gamma<\pi$, at the respective vertices A,B and C.

(see [5, p.284])

Definition 1.4. Let V be a real inner product space and let V_s be the s-ball of V,

$$V_s = \{\mathbf{v}\in V : \|\mathbf{v}\| < s\}$$

where $s>0$ is an arbitrarily fixed constant. Einstein addition $\oplus_E$ is a binary operation in V_s given by the equation

$$\mathbf{u}\oplus_E\mathbf{v} = \frac{1}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{s^2}}\left\{\mathbf{u}+\frac{1}{\gamma_{\mathbf{u}}}\mathbf{v}+\frac{1}{s^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\right\}$$

where $\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1-\frac{\|\mathbf{u}\|}{s^2}}}$ is the gamma factor in V_s, and where $\cdot$ and $\|\cdot\|$ are the inner product and norm that the ball V_s inherits from its space V.

(see [5, p.88])

Theorem 1.1. ***(The Relativistic Law of Gyrocosines)*** *Let ABC be a gyrotriangle in a gyrovector space $(V_s,\oplus,\otimes)$, whose vertices are the points A,B and C of the gyroplane and whose sides are $\mathbf{a}=-B\oplus C, \mathbf{b}=-C\oplus A$, and $\mathbf{c}=-A\oplus B$. Let $a=\|\mathbf{a}\|, b=\|\mathbf{b}\|, c=\|\mathbf{c}\|$ are the side-gyrolengths of the gyrotriangle ABC,* *and whose gyroangles* $\alpha = \angle BAC, \beta = \angle CBA, \gamma = \angle ACB$ *of the gyrotriangle* ABC. *Then,*

$$\gamma_a = \gamma_b \gamma_c (1 - b_s c_s \cos \alpha),$$

where $\gamma_a = \frac{1}{\sqrt{1-a_s^2}}$, *and* $a_s = \frac{a}{s}$.

(see [5, p.542])

Theorem 1.2. ***(The Gyrotriangle Bisector Theorem).*** *Let* ABC *be a gyrotriangle in a Einstein gyrovector space* $(V_s, \oplus, \otimes)$ *with vertices* $A, B, C \in V_s$, *sides* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{V}_s$, *and side gyrolengths* $a, b, c \in (-s, s)$, $\mathbf{a} = \ominus B \oplus C$, $\mathbf{b} = \ominus C \oplus A$, $\mathbf{c} = \ominus A \oplus B$, $a = \|\mathbf{a}\|$, $b = \|\mathbf{b}\|$, $c = \|\mathbf{c}\|$, *and let* D *be a point lying on side* BC *of the gyrotriangle such that* AD *is a bisector of gyroangle* $\angle BAC$. *Then*

$$\frac{\gamma_{|BD|} |BD|}{\gamma_{|CD|} |CD|} = \frac{\gamma_{|AB|} |AB|}{\gamma_{|AC|} |AC|},$$

where $\gamma_v = \frac{1}{\sqrt{1-\frac{v^2}{s^2}}}$.

(see [6, p.151])

For further details we refer to the recent book of A.Ungar [5].

2. Main results

In this sections, we present a proof of the hyperbolic Stewart theorem in the Einstein relativistic velocity model of hyperbolic geometry.

Theorem 2.1. ***(The hyperbolic Stewart theorem)***. *If a point* D *lies between the vertices* A *and* C *of the gyrotriangle* ABC, *then*

$$\gamma_{|AB|} \cdot \gamma_{|DC|} \cdot |DC| + \gamma_{|AC|} \cdot \gamma_{|BD|} \cdot |BD| - \gamma_{|AD|} \cdot \gamma_{|DC|} \cdot \gamma_{|BD|} \cdot [|BD| + |DC|] = 0,$$

where $|DC|, |BD|$, *and* $|BC|$ *noted the gyrolengths of gyrosegments* DC, BD, *and* BC, *respectively.*

Proof. If we use theorem 1.1 in the triangles ABD and BCD respectively (see Figure 1), then

$$\cos \widehat{ADB} = \frac{-\gamma_{|AB|} + \gamma_{|BD|} \cdot \gamma_{|AD|}}{\gamma_{|BD|} \cdot \gamma_{|AD|} \cdot \frac{|BD|}{s} \cdot \frac{|AD|}{s}} \tag{2.1}$$

and

$$\cos \widehat{ADC} = \frac{-\gamma_{|AC|} + \gamma_{|AD|} \cdot \gamma_{|DC|}}{\gamma_{|AD|} \cdot \gamma_{|CD|} \cdot \frac{|AD|}{s} \cdot \frac{|DC|}{s}}. \tag{2.2}$$

Because of $\cos \widehat{ADB} = -\cos \widehat{ADC}$, we obtain

$$\gamma_{|BD|} \cdot \gamma_{|CD|} \cdot \gamma_{|AD|} \cdot |DC| - \gamma_{|AB|} \cdot \gamma_{|DC|} \cdot |DC| = \gamma_{|AC|} \cdot \gamma_{|BD|} \cdot |BD| - \gamma_{|AD|} \cdot \gamma_{|BD|} \cdot \gamma_{|DC|} \cdot |BD|$$

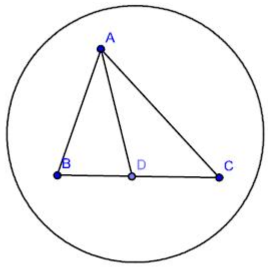

FIGURE 1

or

$$\gamma_{|AB|} \cdot \gamma_{|DC|} \cdot |DC| + \gamma_{|AC|} \cdot \gamma_{|BD|} \cdot |BD| - \gamma_{|AD|} \cdot \gamma_{|DC|} \cdot \gamma_{|BD|} \cdot [|BD| + |DC|] = 0. \quad (2.3)$$

□

Corollary 2.2. ***(Median theorem in hyperbolic geometry).*** *Let ABC be a gyrotriangle, and D is a gyromidpoint of the gyrosegment BC. Then,*

$$\gamma_{|AD|} = \frac{\gamma_{|AB|} + \gamma_{|AC|}}{2 \cdot \gamma_{|DC|}}.$$

Proof. We have

$$|BD| = |DC| \quad (2.4)$$

and

$$\gamma_{|BD|} = \gamma_{|DC|} \quad (2.5)$$

If we use theorem 2.1 and relations (2.4) and (2.5) we get

$$\gamma_{|AD|} = \frac{\gamma_{|AB|} + \gamma_{|AC|}}{2 \cdot \gamma_{|DC|}}. \quad (2.6)$$

□

Corollary 2.3. ***(The gamma factor of an angle bisector).*** *Let ABC be a gyrotriangle, and let D be a point lying on side BC of the gyrotriangle such that AD is a bisector of gyroangle $\angle BAC$. Then*

$$\gamma_{|AD|} = \frac{\gamma_{|AB|} \cdot |DC|}{\gamma_{|BD|} \cdot [|BD| + |DC|]} \cdot \left(1 + \frac{|AB|}{|AC|}\right).$$

Proof. If we use theorem 2.1 in the triangles ABC, then

$$\gamma_{|AB|} \cdot \gamma_{|DC|} \cdot |DC| + \gamma_{|AC|} \cdot \gamma_{|BD|} \cdot |BD| - \gamma_{|AD|} \cdot \gamma_{|DC|} \cdot \gamma_{|BD|} \cdot [|BD| + |DC|] = 0. \quad (2.7)$$

If we use the gyrotriangle bisector theorem, we have

$$\frac{\gamma_{|BD|}\,|BD|}{\gamma_{|CD|}\,|CD|} = \frac{\gamma_{|AB|}\,|AB|}{\gamma_{|AC|}\,|AC|}. \quad (2.8)$$

From (2.7) and (2.8) we get

$$\gamma_{|AD|} = \frac{\gamma_{|AB|} \cdot |DC|}{\gamma_{|BD|} \cdot [|BD| + |DC|]} \cdot \left(1 + \frac{|AB|}{|AC|}\right). \tag{2.9}$$

□

The Einstein relativistic velocity model is another model of hyperbolic geometry. Many of the theorems of Euclidean geometry are relatively similar form in the Einstein relativistic velocity model, Stewart's theorem for gyrotriangle is an example in this respect. We should note that in the Euclidean limit of large s, $s \to \infty$, gamma factor γ_v reduces to 1, so the gyroequality (2.3) reduces to the trivial identity $0 = 0$. Hence, (2.3) has no immediate Euclidean counterpart, thus presenting a disanalogy between hyperbolic and Euclidean geometry.

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