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Some hyperbolic concurrency results in the Poincaré disc

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ABSTRACT. In this paper, we present a proof of some hyperbolic theorems in the Poincaré Model of Hyperbolic Geometry.

1. INTRODUCTION

Hyperbolic geometry appeared in the first half of the 19^{th} century as an attempt to understand Euclid's axiomatic basis of Geometry. It is also known as a type of non-Euclidean Geometry, being in many respects similar to the Euclidean Geometry. Hyperbolic Geometry includes similar concepts as distance and angle. Both these geometries have many results in common but many are different. Several useful models of Hyperbolic Geometry are studied in the literature as, for instance, the Poincaré disc and ball models, the Poincaré half-plane model, and the Beltrami-Klein disc and ball models [8] etc. Following [10] and [11] and earlier discoveries, the Beltrami-Klein model is also known as the Einstein relativistic velocity model.

Here, in this study, we present some applications of the Ceva's theorem in the Poincaré disc model of hyperbolic geometry. The Euclidean version of this well-known theorem states that if three lines from the vertices of a triangle $A_1A_2A_3$ are concurrent at M, and meet the opposide sides at P, Q, R respectively, then $\frac{A_1P}{PA_2} \cdot \frac{A_2R}{RA_3} \cdot \frac{A_3Q}{QA_1} = 1$ [6]. This result has a simple statement but it is of a great interest. We have just mentioned here few different proofs given by N.A.Court [3], D.Grindberg [4], R.Honsberg [5], A.Ungar [9]. In this paper it is also presented the hyperbolic proof of Mathieu's theorem in the Poincaré disc model of the hyperbolic geometry. The classical Mathieu's theorem states that if three lines from the vertices of a triangle are concurrent, their isogonals are also concurrent [7].

We begin with the recall of some basic geometric notions and properties in the Poincaré disc. Let D denote the unit disc in the complex z - plane, i.e.

$$D = \{z \in \mathbb{C} : |z| < 1\}$$

The most general Möbius transformation of D is

$$z \to e^{i\theta} \frac{z_0 + z}{1 + \overline{z_0} z} = e^{i\theta}(z_0 \oplus z),$$

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which induces the Möbius addition $\oplus$ in D, allowing the Möbius transformation of the disc to be viewed as a Möbius left gyro-translation

$$z \to z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0} z}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $z, z_0 \in D$, and $\overline{z_0}$ is the complex conjugate of z_0. Let $Aut(D, \oplus)$ be the automorphism group of the grupoid $(D, \oplus)$. If we define

$$gyr : D \times D \to Aut(D, \oplus), gyr[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\overline{b}}{1 + \overline{a}b},$$

then it is true the gyro-commutative law

$$a \oplus b = gyr[a, b](b \oplus a).$$

A gyro-vector space $(G, \oplus, \otimes)$ is a gyro-commutative gyro-group $(G, \oplus)$ that obeys the following axioms:

(1) $gyr[\mathbf{u}, \mathbf{v}]\mathbf{a} \cdot gyr[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a} \cdot \mathbf{b}$ for all points $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in G$.

(2) G admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $\mathbf{a} \in G$:

(G1) $1 \otimes \mathbf{a} = \mathbf{a}$

(G2) $(r_1 + r_2) \otimes \mathbf{a} = r_1 \otimes \mathbf{a} \oplus r_2 \otimes \mathbf{a}$

(G3) $(r_1 r_2) \otimes \mathbf{a} = r_1 \otimes (r_2 \otimes \mathbf{a})$

(G4) $\frac{|r| \otimes \mathbf{a}}{\|r \otimes \mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$

(G5) $gyr[\mathbf{u}, \mathbf{v}](r \otimes \mathbf{a}) = r \otimes gyr[\mathbf{u}, \mathbf{v}]\mathbf{a}$

(G6) $gyr[r_1 \otimes \mathbf{v}, r_1 \otimes \mathbf{v}] = 1$

(3) Real vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of one-dimensional "vectors"

$$\|G\| = \{\pm \|\mathbf{a}\| : \mathbf{a} \in G\} \subset \mathbb{R}$$

with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in G$,

(G7) $\|r \otimes \mathbf{a}\| = |r| \otimes \|\mathbf{a}\|$

(G8) $\|\mathbf{a} \oplus \mathbf{b}\| \leq \|\mathbf{a}\| \oplus \|\mathbf{b}\|$

Theorem 1.1. (***The law of gyrosines in Möbius gyrovector spaces***). *Let ABC be a gyrotriangle in a Möbius gyrovector space $(V_s, \oplus, \otimes)$ with vertices $A, B, C \in V_s$, sides $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{V}_s$, and side gyrolengths $a, b, c \in (-s, s)$, $\mathbf{a} = \ominus B \oplus C$, $\mathbf{b} = \ominus C \oplus A$, $\mathbf{c} = \ominus A \oplus B$, $a = \|\mathbf{a}\|$, $b = \|\mathbf{b}\|$, $c = \|\mathbf{c}\|$, and with gyroangles α, β, and γ at the vertices A, B, and C. Then $\frac{a_\gamma}{\sin \alpha} = \frac{b_\gamma}{\sin \beta} = \frac{c_\gamma}{\sin \gamma}$, where $v_\gamma = \frac{v}{1 - \frac{v^2}{s^2}}$* [9, *p. 267*].

Definition 1.1. *The hyperbolic distance function in D is defined by the equation*

$$d(a, b) = |a \ominus b| = \left| \frac{a - b}{1 - \overline{a}b} \right|.$$

Here, $a \ominus b = a \oplus (-b)$, for $a, b \in D$.

For further details we refer to the recent book of A.Ungar [10].

Theorem 1.2. (***The Gyrotriangle Bisector Theorem***). *Let ABC be a gyrotriangle in a Möbius gyrovector space $(V_s, \oplus, \otimes)$ with vertices $A, B, C \in V_s$, sides $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{V}_s$, and side gyrolengths $a, b, c \in (-s, s)$, $\mathbf{a} = \ominus B \oplus C$, $\mathbf{b} = \ominus C \oplus A$, $\mathbf{c} = \ominus A \oplus B$, $a = \|\mathbf{a}\|$, $b = \|\mathbf{b}\|$, $c = \|\mathbf{c}\|$, and let D be a point lying on side BC of the gyrotriangle so that AD is a bisector of gyroangle $\angle BAC$. Then*

$$\frac{(DB)_\gamma}{(DC)_\gamma} = \frac{(AB)_\gamma}{(AC)_\gamma},$$

where $v_\gamma = \frac{v}{1-\frac{v^2}{s^2}}$.

(See [2])

Theorem 1.3. (***The Ceva's Theorem for the Hyperbolic Gyrotriangle***). *If M is a point not on any side of a gyrotriangle $A_1A_2A_3$ such that A_3M and A_1A_2 meet in P, A_2M and A_3A_1 in Q, and A_1M and A_2A_3 meet in R, then*

$$\frac{(A_1P)_\gamma}{(A_2P)_\gamma} \cdot \frac{(A_2R)_\gamma}{(A_3R)_\gamma} \cdot \frac{(A_3Q)_\gamma}{(A_1Q)_\gamma} = 1$$

(See [1])

Theorem 1.4. (***Converse of Ceva's Theorem for Hyperbolic Gyrotriangle***). *If P lies on the gyroline A_1A_2, R on A_2A_3, and Q on A_3A_1 so that*

$$\frac{(A_1P)_\gamma}{(A_2P)_\gamma} \cdot \frac{(A_2R)_\gamma}{(A_3R)_\gamma} \cdot \frac{(A_3Q)_\gamma}{(A_1Q)_\gamma} = 1,$$

and two of the gyrolines A_1R, A_2Q and A_3P meet, then all three are concurrent.

(See [1])

2. Main results

In this section we prove some applications of Ceva's theorem in the Poincaré disc model of hyperbolic geometry

The Euclidean case of the following theorem seems to be done by Patrascu.

Theorem 2.5. *Let D be a point on the gyroside BC of a gyrotriangle ABC, and let E and F be the points lying on sides CA and AB of the gyrotriangle ABC so that DE is a bisector of the gyroangle $\angle ADC$, and DF is a bisector of the gyroangle $\angle ADB$, then the gyrolines AD, BE, and CF are concurrent.*

Proof. The bisector theorem (See Theorem 1.2.), gives for the gyrotriangles ABD and ACD (See Figure 1) respectively,

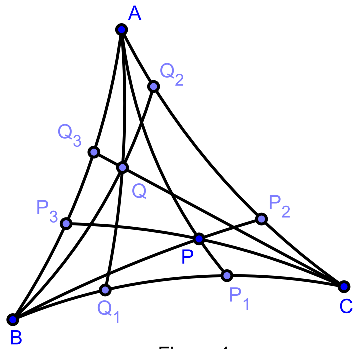

Figure 1

$$\frac{(FB)_\gamma}{(FA)_\gamma} = \frac{(BD)_\gamma}{(AD)_\gamma} \tag{1}$$

and

$$\frac{(EA)_\gamma}{(EC)_\gamma} = \frac{(AD)_\gamma}{(CD)_\gamma}. \tag{2}$$

Then,

$$\frac{(FB)_\gamma}{(FA)_\gamma} \cdot \frac{(EA)_\gamma}{(EC)_\gamma} \cdot \frac{(DC)_\gamma}{(DB)_\gamma} = \frac{(BD)_\gamma}{(AD)_\gamma} \cdot \frac{(AD)_\gamma}{(CD)_\gamma} \cdot \frac{(DC)_\gamma}{(DB)_\gamma} = 1. \tag{3}$$

Because the gyrolines AD, BE, and CF all lie within the gyrotriangle, so both must meet. Thus, by the Converse of Ceva's Theorem, the gyrolines AD, BE, and CF are concurrent. ∎

Theorem 2.6. ***(The Ţiţeica's Theorem for the Hyperbolic Gyrotriangle).*** *Let $A_1B_1C_1$ be the Cevian gyrotriangle of the gyropoint P with respect to the gyrotriangle ABC, and let l be a gyroline not through any vertex of a gyrotriangle ABC so that l meets the gyrosides BC, CA, and AB in the points A_2, B_2, and C_2, respectively. If the gyrolines B_1C_2 and BC meet in the gyropoint A_3, the gyrolines C_1A_2 and CA meet in the gyropoint B_3, and the gyrolines A_1B_2 and AB meet in the gyropoint C_3, then the gyrolines AA_3, BB_3, and CC_3 are concurrent.*

Proof. If we use the Menelaus theorem in the gyrotriangle ABC (See Figure 2) for the gyrolines $A_3B_1C_2$, $B_3C_1A_2$, $C_3A_1B_2$, and $A_2B_2C_2$, respectively, we have

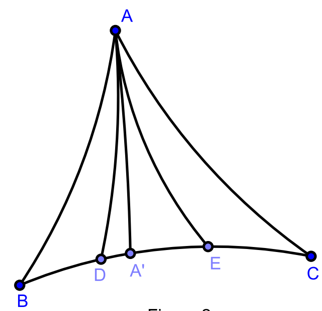

Figure 2

$$\frac{(A_3C)_\gamma}{(A_3B)_\gamma} = \frac{(B_1C)_\gamma}{(B_1A)_\gamma} \cdot \frac{(C_2A)_\gamma}{(C_2B)_\gamma}, \tag{4}$$

and

$$\frac{(B_3A)_\gamma}{(B_3C)_\gamma} = \frac{(C_1A)_\gamma}{(C_1B)_\gamma} \cdot \frac{(A_2B)_\gamma}{(A_2C)_\gamma}, \tag{5}$$

and

$$\frac{(C_3B)_\gamma}{(C_3A)_\gamma} = \frac{(A_1B)_\gamma}{(A_1C)_\gamma} \cdot \frac{(B_2C)_\gamma}{(B_2A)_\gamma}, \tag{6}$$

and

$$\frac{(A_2B)_\gamma}{(A_2C)_\gamma} \cdot \frac{(B_2C)_\gamma}{(B_2A)_\gamma} \cdot \frac{(C_2A)_\gamma}{(C_2B)_\gamma} = 1. \tag{7}$$

If we use the Ceva theorem in the gyrotriangle ABC we get

$$\frac{(A_1B)_\gamma}{(A_1C)_\gamma} \cdot \frac{(B_1C)_\gamma}{(B_1A)_\gamma} \cdot \frac{(C_1A)_\gamma}{(C_1B)_\gamma} = 1. \tag{8}$$

From (4), (5), (6), (7), and (8), we get

$$\frac{(A_3C)_\gamma}{(A_3B)_\gamma} \cdot \frac{(B_3A)_\gamma}{(B_3C)_\gamma} \cdot \frac{(C_3B)_\gamma}{(C_3A)_\gamma} =$$

$$\left(\frac{(B_1C)_\gamma}{(B_1A)_\gamma} \cdot \frac{(C_1A)_\gamma}{(C_1B)_\gamma} \cdot \frac{(A_1B)_\gamma}{(A_1C)_\gamma}\right) \cdot \left(\frac{(C_2A)_\gamma}{(C_2B)_\gamma} \cdot \frac{(A_2B)_\gamma}{(A_2C)_\gamma} \cdot \frac{(B_2C)_\gamma}{(B_2A)_\gamma}\right) = 1. \tag{9}$$

Thus, by the Converse of Ceva's Theorem, the gyrolines AA_3, BB_3, and CC_3 are concurrent. ∎

Next, we demonstrate Mathieu's theorem for hyperbolic triangle.

Theorem 2.7. ***(Hyperbolic Mathieu's Theorem)*** *If three gyrolines from a gyrotriangle* ABC*, and concurrent at* P*, meet the opposite gyrosides at* P_1, P_2, P_3 *respectively, and the gyrolines* $AQ_1, BQ_{2,}$ *and* CQ_3 *are their isogonal gyrolines, and two of the gyrolines* $AQ_1, BQ_{2,}$ *and* CQ_3 *meet, then all three are also concurrent.*

Proof. Let a, b, c denote the sidelengths in the standard order. We set $\angle BAQ_1 = \angle P_1AC = \theta, \angle BAC = \alpha, \angle CBA = \beta$, $\angle ACB = \gamma, \angle AQ_1B = \epsilon_1, \angle AQ_1C = \epsilon_2, \angle AP_1B = \lambda_1, \angle AP_1C = \lambda_2$

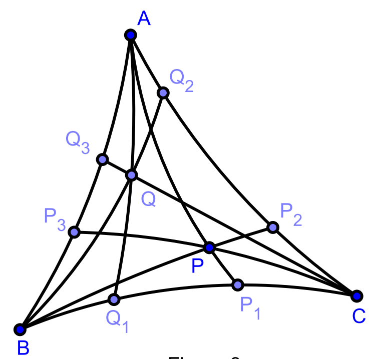

Figure 3

If we use the Ceva's theorem in the gyrotriangle ABC (See Theorem 1.3., Figure 3), then

$$\frac{(CP_1)_\gamma}{(BP_1)_\gamma} \cdot \frac{(AP_2)_\gamma}{(CP_2)_\gamma} \cdot \frac{(BP_3)_\gamma}{(AP_3)_\gamma} = 1. \tag{10}$$

If we use the law of gyrosine in the gyrotriangles $ABQ_1, ACQ_1, ACP_1, ABP_1$ respectively (see Theorem 1.1.), then

$$\frac{\sin\theta}{(BQ_1)_\gamma} = \frac{\sin\epsilon_1}{(AB)_\gamma}, \tag{11}$$

$$\frac{\sin(A-\theta)}{(CQ_1)_\gamma} = \frac{\sin\epsilon_2}{(AC)_\gamma}, \tag{12}$$

$$\frac{\sin\theta}{(CP_1)_\gamma} = \frac{\sin\lambda_2}{(AC)_\gamma}, \tag{13}$$

$$\frac{\sin(A-\theta)}{(BP_1)_\gamma} = \frac{\sin\lambda_1}{(AB)_\gamma}. \tag{14}$$

If ratios the equations (11) and (12) among themselves, respectively, and because $\sin(\pi-\theta) = \sin\theta, \forall x \in \mathbb{R}$, then

$$\frac{\sin\theta}{\sin(A-\theta)} \cdot \frac{(CQ_1)_\gamma}{(BQ_1)_\gamma} = \frac{(AC)_\gamma}{(AB)_\gamma}. \tag{15}$$

If ratios the equations (13) and (14) among themselves, respectively, then

$$\frac{\sin\theta}{\sin(A-\theta)} \cdot \frac{(BP_1)_\gamma}{(CP_1)_\gamma} = \frac{(AB)_\gamma}{(AC)_\gamma}. \tag{16}$$

If ratios the equations (15) and (16) among themselves, respectively, then

$$\frac{(CQ_1)_\gamma}{(BQ_1)_\gamma}\cdot\frac{(CP_1)_\gamma}{(BP_1)_\gamma}=\left(\frac{(AC)_\gamma}{(AB)_\gamma}\right)^2. \tag{17}$$

Similarly,

$$\frac{(AQ_2)_\gamma}{(CQ_2)_\gamma}\cdot\frac{(AP_2)_\gamma}{(CP_2)_\gamma}=\left(\frac{(AB)_\gamma}{(BC)_\gamma}\right)^2, \tag{18}$$

and

$$\frac{(BQ_3)_\gamma}{(AQ_3)_\gamma}\cdot\frac{(BP_3)_\gamma}{(AP_3)_\gamma}=\left(\frac{(BC)_\gamma}{(AC)_\gamma}\right)^2. \tag{19}$$

Multiplying the relations (17), (18), and (19), then we get

$$\left(\frac{(BQ_3)_\gamma}{(AQ_3)_\gamma}\cdot\frac{(AQ_2)_\gamma}{(CQ_2)_\gamma}\cdot\frac{(CQ_1)_\gamma}{(BQ_1)_\gamma}\right)\cdot$$

$$\left(\frac{(CP_1)_\gamma}{(BP_1)_\gamma}\cdot\frac{(AP_2)_\gamma}{(CP_2)_\gamma}\cdot\frac{(BP_3)_\gamma}{(AP_3)_\gamma}\right)=1. \tag{20}$$

From (10) and (20) we obtain

$$\frac{(BQ_3)_\gamma}{(AQ_3)_\gamma}\cdot\frac{(AQ_2)_\gamma}{(CQ_2)_\gamma}\cdot\frac{(CQ_1)_\gamma}{(BQ_1)_\gamma}=1, \tag{21}$$

and by Theorem 1.4. we obtain that the gyrolines $AQ_1, BQ_2,$ and CQ_3 are concurrent in a point Q, called the isogonal conjugate of P. ∎

Corollary 2.1. *The incenter I of a gyrotriangle is its own isogonal conjugate of I.*

Definition 2.2. *The symmetric of the median with respect to the internal bisector issued from the same vertex is called symmedian.*

From Mathieu's theorem result that the symmedians of the triangle ABC are concurrent. The point of concurrence of symmedians is Lemoine's point of triangle ABC.

Corollary 2.2. *The isogonal conjugate of centroid of triangle ABC is Lemoine's point of triangle ABC.*

Theorem 2.8. *Let AA' be a interior bisector of a gyroangle BAC of a gyrotriangle ABC, and let D be a gyropoint on the gyroside BC. If AE is the isogonal gyroline of AD, then*

$$\frac{(BD)_\gamma}{(DA')_\gamma}\cdot\frac{(A'E)_\gamma}{(EC)_\gamma}=\frac{(AB)_\gamma}{(AC)_\gamma}$$

Proof. We set $\angle BAD=\angle EAC=\alpha, \angle DAA'=\angle A'AE=\beta, \angle BDA=\theta, \angle AEC=\lambda$ (see Figure 4).

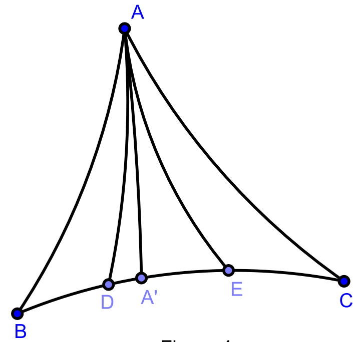

Figure 4

If we use the law of gyrosine in the gyrotriangles $ABD, DAA', AEC, AA'E$ respectively (see Theorem 1.1.), then

$$\frac{\sin\alpha}{(BD)_\gamma} = \frac{\sin\theta}{(AB)_\gamma}, \tag{22}$$

$$\frac{\sin\beta}{(DA')_\gamma} = \frac{\sin(\pi-\theta)}{(AA')_\gamma}, \tag{23}$$

$$\frac{\sin\alpha}{(EC)_\gamma} = \frac{\sin\lambda}{(AC)_\gamma}, \tag{24}$$

$$\frac{\sin\beta}{(A'E)_\gamma} = \frac{\sin(\pi-\lambda)}{(AA')_\gamma}. \tag{25}$$

If ratios the equations (22) and (23) among themselves, respectively, and because $\sin(\pi-\theta) = \sin\theta$, then

$$\frac{\sin\beta}{\sin\alpha}\cdot\frac{(BD)_\gamma}{(DA')_\gamma} = \frac{(AB)_\gamma}{(AA')_\gamma}. \tag{26}$$

If ratios the equations (24) and (25) among themselves, respectively, then

$$\frac{\sin\beta}{\sin\alpha}\cdot\frac{(EC)_\gamma}{(A'E)_\gamma} = \frac{(AC)_\gamma}{(AA')_\gamma}. \tag{27}$$

If ratios the equations (26) and (27) among themselves, respectively, then

$$\frac{(BD)_\gamma}{(DA')_\gamma}\cdot\frac{(A'E)_\gamma}{(EC)_\gamma} = \frac{(AB)_\gamma}{(AC)_\gamma}. \tag{28}$$

∎

Corollary 2.3. *Let AA' be a interior bisector of a gyroangle BAC of a gyrotriangle ABC, and let D be a gyropoint on the gyroside BC. If AE is the isogonal gyroline of AD, then*

$$\frac{(AB)_\gamma}{(AC)_\gamma}\cdot\frac{(AD)_\gamma}{(AE)_\gamma}\cdot\frac{(EC)_\gamma}{(BD)_\gamma} = 1.$$

Proof. If we use the bisector theorem in the gyrotriangles DAE (see Theorem 1.2.), then

$$\frac{(A'E)_\gamma}{(DA')_\gamma} = \frac{(AE)_\gamma}{(AD)_\gamma}. \tag{29}$$

If we use Theorem 2.8 and the relations 29, we get to the conclusion. ∎

Remark 2.1. *If we use the bisector theorem in the gyrotriangles DAE, then $\frac{(AB)_\gamma}{(AC)_\gamma} = \frac{(A'B)_\gamma}{(A'C)_\gamma}$ and the relation (28) becomes*

$$\frac{(BD)_\gamma}{(DA')_\gamma} \cdot \frac{(A'E)_\gamma}{(EC)_\gamma} \cdot \frac{(A'C)_\gamma}{(A'B)_\gamma} = 1.$$

Corollary 2.4. *Let ABC be a isosceles gyrotriangle, and let D be a gyropoint on the gyroside BC. If AE is the isogonal gyroline of AD, then*

$$\frac{(AD)_\gamma}{(AE)_\gamma} = \frac{(BD)_\gamma}{(EC)_\gamma}.$$

Proof. Because $AB = AC$, result that $(AB)_\gamma = (AC)_\gamma$. Now, from the relation (28) we obtain the conclusion. ∎

Many of the theorems of the Euclidean geometry are relatively similar form in the Poincaré model of hyperbolic geometry, Mathieu's theorem is an example in this respect. In the Euclidean limit of large s, $s \to \infty$, v_γ reduces to v, so Mathieu's theorem for the hyperbolic triangle reduces to the Mathieu's theorem of the euclidian geometry.

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