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Research Paper

New Inequalities on Hyperbolic Triangles

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Abstract: *In this paper we will analyze some new inequalities on hyperbolic triangle.*

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1. Introduction:

In paper [1] are presented some hyperbolic Huygens type inequalities, such as:

$$\frac{\sinh x}{x} < \frac{2}{3} + \frac{1}{3}\cosh x \tag{1}$$

for all $x \in (0, \infty)$. In this paper we will use this inequality to generate some new inequalities in hyperbolic triangle. For more on topic of hyperbolic triangle inequalities see [2],[3].

2. Main Results:

Theorem 2.1 *In any hyperbolic triangle the following inequalities holds:*

$$\cosh a + \cosh b + \cosh c \le \alpha + 2\left(\sinh^2 \frac{a}{2} + \sinh^2 \frac{b}{2} + \sinh^2 \frac{c}{2}\right) \tag{2}$$

$$\cosh a + \cosh b \le 2\left(\cosh^2 \frac{a}{2} + \cosh^2 \frac{b}{2}\right) \tag{3}$$

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$$\sinh a+\sinh b+\sinh c<$$
$$<4\left(\sinh\frac{a}{2}+\sinh\frac{b}{2}+\sinh\frac{c}{2}+\sinh\frac{a}{2}\sinh^2\frac{a}{4}+\sinh\frac{b}{2}\sinh^2\frac{b}{4}+\sinh\frac{c}{2}\sinh^2\frac{c}{4}\right) \tag{4}$$

where $\alpha\geq 3$.

Proof : For the first inequality, we use the equality

$$\cosh x=1+2\sinh^2\frac{x}{2}$$

for all $x\in\mathbb{R}$. One obtain: $\cosh a+\cosh b+\cosh c=3+2\left(\sinh^2\frac{a}{2}+\sinh^2\frac{b}{2}+\sinh^2\frac{c}{2}\right)$ (5)

So, the first inequality hold. Now we will proof the second inequality from theorem 2.1

$$\cosh a+\cosh b=2\left(1+\sinh^2\frac{a}{2}+\sinh^2\frac{b}{2}\right)\leq$$
$$\leq 2\left(\cosh^2\frac{a}{2}+\sinh^2\frac{b}{2}\right)+2\leq 2\left(\cosh^2\frac{a}{2}+\cosh^2\frac{b}{2}\right)$$

So, the second inequality hold. For the last inequality, let us consider the inequalities

$$\cosh a<2\left(1+\sinh^2\frac{a}{2}\right) \tag{6}$$
$$\cosh b<2\left(1+\sinh^2\frac{b}{2}\right) \tag{7}$$
$$\cosh c<2\left(1+\sinh^2\frac{c}{2}\right) \tag{8}$$

Using this three inequalities, and also the equality

$$\sinh a+\sinh b+\sinh c=2\sinh\frac{a}{2}\cosh\frac{a}{2}+2\sinh\frac{b}{2}\cosh\frac{b}{2}+2\sinh\frac{c}{2}\cosh\frac{c}{2}\quad ,$$

the proof of this theorem is done.

Theorem 2.2 *The function* $\cosh x$ *is a convex function, so, the following inequality hold:*

$$3\cosh\frac{a+b+c}{3}\leq\frac{\cosh a+\cosh b+\cosh c}{3} \tag{9}$$

Observation 2.3 *From Theorem 2.1 and Theorem 2.2, one obtain:*

$$3\cosh\frac{a+b+c}{3}\leq\cosh a+\cosh b+\cosh c\leq\alpha+2\left(\sinh^2\frac{a}{2}+\sinh^2\frac{b}{2}+\sinh^2\frac{c}{2}\right) \tag{10}$$

where $\alpha\geq 3$.

Corollary 2.4 *In any hyperbolic triangle we have:*

$$\sinh^2\frac{a}{2}+\sinh^2\frac{b}{2}+\sinh^2\frac{c}{2}\geq\frac{3}{2}\left(1-\cosh\frac{a+b+c}{3}\right)$$

Proof : Using relation $\cosh x=1+2\sinh^2\frac{x}{2}$, for $x\in\mathrm{R}$ and Theorem 2.2, one obtain:

$$3\cosh\frac{a+b+c}{3}\leq 3+2\left(\sinh^2\frac{a}{2}+\sinh^2\frac{b}{2}+\sinh^2\frac{c}{2}\right),$$

and the conclusion follows.

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Corollary 2.5 *In any hyperbolic triangle is true the following inequality:*

$$\cosh\frac{c}{2} < \cosh^2\frac{a}{2} + \cosh^2\frac{b}{2}$$

Proof : From convexity of cosh function and from triangle inequality, one obtain:

$$\frac{\cosh a + \cosh b}{2} \geq \cosh\frac{a+b}{2} > \cosh\frac{c}{2}$$

From this inequality and inequality (3) we get:

$$2\cosh\frac{c}{2} < \cosh a + \cosh b \leq 2\left(\cosh^2\frac{a}{2} + \cosh^2\frac{b}{2}\right)$$

and the conclusion follows.

Lemma 2.6 *The following inequality hold:*

$$\frac{\sinh a}{a} + \frac{\sinh b}{b} < \frac{4}{3} + \frac{2}{3}\left(\cosh^2\frac{a}{2} + \cosh^2\frac{b}{2}\right) \tag{11}$$

Proof : Using inequality (1) and (2), one obtain:

$$\frac{\sinh a}{a} < \frac{2}{3} + \frac{1}{3}(\cosh a) \tag{12}$$

$$\frac{\sinh b}{b} < \frac{2}{3} + \frac{1}{3}(\cosh b) \tag{13}$$

Using inequalities (12) and (13), one obtain:

$$\frac{\sinh a}{a} + \frac{\sinh b}{b} < \frac{4}{3} + \frac{1}{3}(\cosh a + \cosh b) \leq \frac{4}{3} + \frac{2}{3}\left(\cosh^2\frac{a}{2} + \cosh^2\frac{b}{2}\right)$$

which complete the proof of the lemma 2.6.

Lemma 2.7 *The following inequality hold:*

$$\frac{\sinh a}{a} + \frac{\sinh b}{b} + \frac{\sinh c}{c} < \frac{2}{3}\left(5 + \sinh^2\frac{a}{2} + \sinh^2\frac{b}{2} + \sinh^2\frac{c}{2}\right)$$

Proof : From ineqality (2), we get:

$$\cosh a + \cosh b + \cosh c \leq 4 + 2\left(\sinh^2\frac{a}{2} + \sinh^2\frac{b}{2} + \sinh^2\frac{c}{2}\right)$$

and using three times inequality (12), one obtain:

$$\frac{\sinh a}{a} + \frac{\sinh b}{b} + \frac{\sinh c}{c} < \frac{6}{3} + \frac{1}{3}\left(4 + 2\left(\sinh^2\frac{a}{2} + \sinh^2\frac{b}{2} + \sinh^2\frac{c}{2}\right)\right) \tag{14}$$

so, we obtain:

$$\frac{\sinh a}{a} + \frac{\sinh b}{b} + \frac{\sinh c}{c} < \frac{2}{3}\left(5 + \sinh^2\frac{a}{2} + \sinh^2\frac{b}{2} + \sinh^2\frac{c}{2}\right)$$

Lemma 2.8 *The following inequality hold:*

$$6\sinh\frac{a+b+c}{6} \leq \sinh a + \sinh b + \sinh c \tag{15}$$

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Proof: We will use the following inequalities:

$$\sinh x = 2\sinh\frac{x}{2}\cosh\frac{x}{2} \geq 2\sinh\frac{x}{2}$$

$$2\left(\sinh\frac{a}{2}+\sinh\frac{b}{2}+\sinh\frac{c}{2}\right) \leq \sinh a + \sinh b + \sinh c \quad (16)$$

Also, using the convexity of sinh function, one obtain:

$$\sinh\frac{a+b+c}{6} = \sinh\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{3} \leq \frac{\sinh\frac{a}{2}+\sinh\frac{b}{2}+\sinh\frac{c}{2}}{3} \quad (17)$$

From (16) and (17), we obtain the proof of corollary 2.8.

Corollary 2.9 *In any hyperbolic triangle, hold the following inequality:*

$$\sinh\frac{a+b+c}{6} <$$

$$< \frac{2}{3}\left(\sinh\frac{a}{2}+\sinh\frac{b}{2}+\sinh\frac{c}{2}+\sinh\frac{a}{2}\sinh^2\frac{a}{4}+\sinh\frac{b}{2}\sinh^2\frac{b}{4}+\sinh\frac{c}{2}\sinh^2\frac{c}{4}\right) \quad (18)$$

Proof: Using the inequalities (15) and (4), we obtain immediately the assertion of this corollary.

Corollary 2.10 *In any hyperbolic triangle, the following inequality hold:*

$$\sinh\frac{a}{3} < \frac{2}{3}\left(\sinh\frac{a}{2}+\sinh\frac{b}{2}+\sinh\frac{c}{2}+\sinh\frac{a}{2}\sinh^2\frac{a}{4}+\sinh\frac{b}{2}\sinh^2\frac{b}{4}+\sinh\frac{c}{2}\sinh^2\frac{c}{4}\right) \quad (19)$$

Proof: Using the triangle inequality $b+c>a$**,** we get: $a+b+c>2a$, and then $\frac{a+b+c}{6} > \frac{a}{3}$**.** Taking into account that the sine function is an increasing function, one obtain $\sinh\frac{a+b+c}{6} > \sinh\frac{a}{3}$ (20)

Using inequality (20) and the corollary 2.9, we get the assertion of the corollary**.**

References

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