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INEQUALITIES WITH GUDERMANNIANS IN HYPERBOLIC TRIANGLE

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ABSTRACT. In paper [3] we presented some inequalities for hyperbolic triangles. In this paper we will use this inequalities to obtain some new inequalities in hyperblolic triangle using the gudermannian function.

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1. INTRODUCTION

The gudermannian function relates the circular functions and the hyperbolic functions without using complex numbers.
The gudermannian function is defined by:

$$gd\,x = \int_0^x \frac{dt}{\cosh t} = \arcsin(\tanh x) = 2\arctan(e^x) - \frac{1}{2}\pi \tag{1}$$

The inverse of the gudermannian function is defined on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$ is given by

$$gd^{\,-1}x = \int_0^x \frac{dt}{\cos t} = \ln\left|\frac{1+\sin x}{\cos x}\right| \tag{2}$$

Some properties of gudermannian function are:

$$\sin(gd\,x) = \tanh x \tag{3}$$

$$\sec(gd\,x) = \cosh x \tag{4}$$

$$\tan(gd\,x) = \sinh x \tag{5}$$

$$\csc(gd\,x) = \coth x \tag{6}$$

For more details on gudermannian function please see [1] and [2].
Next, let us consider the expansion in Mac Laurin series of the following functions:

$$gd\,x = x - \frac{1}{3}x^3 + \frac{1}{24}x^5 - \frac{61}{5040}x^7 + ... \tag{7}$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + ... \tag{8}$$

$$\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + ... \tag{9}$$

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In paper [1], we obtain the following results:

$$\cosh a + \cosh b \leq 2\left(\cosh^2\frac{a}{2} + \cosh^2\frac{b}{2}\right) \tag{10}$$

$$3\cosh\frac{a+b+c}{3} \leq \frac{\cosh a + \cosh b + \cosh c}{3} \tag{11}$$

$$\cosh c \leq \cosh^2\frac{a}{2} + \cosh^2\frac{b}{2} \tag{12}$$

$$\frac{\sinh \mathrm{a}}{\mathrm{a}} + \frac{\sinh \mathrm{b}}{\mathrm{b}} < \frac{4}{3} + \frac{2}{3}\left(\cosh^2\frac{\mathrm{a}}{2} + \cosh^2\frac{\mathrm{b}}{2}\right) \tag{13}$$

$$6\sinh\frac{a+b+c}{6} \leq \sinh a + \sinh b + \sinh c \tag{14}$$

In this paper we will work with this relations ((10)-(14)) and using the first terms of the expansions (7), (8) and (9) we will obtain new inequalities for hyperbolic triangle using the gudermannian function.

2. Main Results

Theorem 2.1 *In any hyperbolic triangle, the following inequality hold:*

$$(gd\, a)^2 + (gd\, b)^2 \leq 2\left(\cosh^2\frac{a}{2} + \sinh^2\frac{b}{2}\right) \tag{15}$$

Proof. Using (4), the inequation (10) became:

$$\sec(gd\, a) + \sec(gd\, b) \leq 2\left(\cosh^2\frac{a}{2} + \cosh^2\frac{b}{2}\right) \tag{16}$$

In relation (9) if we replace x with the function $gd\, x$ then one obtains:

$$\sec(gd\, x) = 1 + \frac{1}{2}(gd\, x)^2 + \frac{5}{24}(gd\, x)^4 + \frac{61}{720}(gd\, x)^6 + ... \tag{17}$$

Using (16) and the first two terms of (17), we get:

$$1 + \frac{1}{2}(gd\, a)^2 + 1 + \frac{1}{2}(gd\, b)^2 \leq 2\left(\cosh^2\frac{a}{2} + \cosh^2\frac{b}{2}\right) \tag{18}$$

After easy computations in inequation (18) we get:

$$\frac{1}{2}(gd\, a)^2 + \frac{1}{2}(gd\, b)^2 \leq 2\left(\cosh^2\frac{a}{2} + \cosh^2\frac{b}{2} - 1\right)$$

And then, because $\cosh^2\frac{b}{2} - 1 = \sinh^2\frac{b}{2}$, one obtains the assertion of the theorem . ∎

Theorem 2.2 *In any hyperbolic triangle, the following inequality hold:*

$$\left(gd\,\frac{a+b+c}{3}\right)^2 \leq \frac{2(\cosh a + \cosh b + \cosh c) - 18}{9} \tag{19}$$

Proof. Using (4), the inequation (11) became:

$$\left(\sec\left(gd\,\frac{a+b+c}{3}\right)\right) \leq \frac{\cosh a+\cosh b+\cosh c}{9} \tag{20}$$

Using first two terms of (17) one obtains:

$$1+\frac{1}{2}\left(gd\,\frac{a+b+c}{3}\right)^2 \leq \frac{\cosh a+\cosh b+\cosh c}{9} \tag{21}$$

After easy computations in inequation (21) we obtains:

$$\frac{1}{2}\left(gd\,\frac{a+b+c}{3}\right)^2 \leq \frac{\cosh a+\cosh b+\cosh c}{9}-1 \tag{22}$$

Multiplying relation (22) with 2, one obtains the assertion of the theorem . ∎

Theorem 2.3 *In any hyperbolic triangle, the following inequality hold:*

$$(gd\,c)^2 \leq 2\left(\cosh^2\frac{a}{2}+\sinh^2\frac{b}{2}\right) \tag{23}$$

Proof. Using (4), the inequation (12) became:

$$\sec(gd\,c) \leq \cosh^2\frac{a}{2}+\cosh^2\frac{b}{2} \tag{24}$$

Taking first two terms of (17) and replacing in (24), one obtains:

$$1+\frac{1}{2}(gd\,c)^2 \leq \cosh^2\frac{a}{2}+\cosh^2\frac{b}{2} \tag{25}$$

We know that $\cosh^2\frac{b}{2}-1=\sinh^2\frac{b}{2}$.
Using this and also after easy computations one obtains the inequality (23) and in this way the proof of theorem is done.
∎

Theorem 2.4 *In any hyperbolic triangle, the following inequality hold:*

$$\frac{gd\,a}{a}+\frac{gd\,b}{b}+\frac{(gd\,a)^3}{3a}+\frac{(gd\,b)^3}{3b} \leq \frac{4}{3}+\frac{2}{3}\left(\cosh^2\frac{a}{2}+\cosh^2\frac{b}{2}\right) \tag{26}$$

Proof. Using (5), the inequation (13) became:

$$\frac{\tan(gd\,a)}{a}+\frac{\tan(gd\,b)}{b} < \frac{4}{3}+\frac{2}{3}\left(\cosh^2\frac{a}{2}+\cosh^2\frac{b}{2}\right) \tag{27}$$

Using the first two terms of the expansion in Mac Laurin functions

$$\tan(gd\,x)=gd\,x+\frac{1}{3}(gd\,x)^3+\frac{2}{15}(gd\,x)^5+\frac{17}{315}(gd\,x)^7+... \tag{28}$$

and replacing in (27), one obtains:

$$\frac{gd\,a}{a}+\frac{gd\,b}{b}+\frac{(gd\,a)^3}{3a}+\frac{(gd\,b)^3}{3b} \leq \frac{4}{3}+\frac{2}{3}\left(\cosh^2\frac{a}{2}+\cosh^2\frac{b}{2}\right) \tag{29}$$

After easy computation in (28), we obtains the inequality (26), so the theorem is proved. ∎

Theorem 2.5 *In any hyperbolic triangle, the following inequality hold:*

$$6\left(gd\,\frac{a+b+c}{6}\right)+2\left(gd\,\frac{a+b+c}{6}\right)^{3}\leq \sinh a+\sinh b+\sinh c \tag{30}$$

Proof. Using (14) and (28), we get:

$$6\left(\left(gd\,\frac{a+b+c}{6}\right)+\frac{1}{3}\left(gd\,\frac{a+b+c}{6}\right)^{3}\right)\leq \sinh a+\sinh b+\sinh c \tag{31}$$

After easy computation in (31) we get the assertion of the theorem. ∎

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