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K-BÉZIER TYPE CURVES GENERATED BY A KING TYPE OPERATOR

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ABSTRACT. The King type operators were introduced by J.P.King in paper ([2]) in 2003. The King type operators generalized the classical Bernstein operators. A special King type operator was defined in paper ([1]). In this paper, we study a new class of Bézier curves generated by the polynomials used in construction of the King type operator used in paper ([2]).
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1. INTRODUCTION

First, let's recall some classical results from paper ([1]):
Let $m_0 \in N$ and $m_0 > 2$. For the function $f : [0,1] \to \mathbb{R}$, in paper ([1]), is defined the sequence of operators $(B_m^* f)_{m \geq m_0}$, by:

$$(B_m^* f)(x) = \frac{(m-1)x}{mx-1}\left(1-\frac{1}{m}\right)^{-m}\sum_{k=0}^{m}\binom{m}{k}(1-x)^{m-k}\left(x-\frac{1}{m}\right)^k f\left(\frac{k}{m}\right)$$

for any $m \geq m_0$, and any $x \in \left[\frac{1}{m_0-1}, 1\right]$. For fixed m_0, from $m \geq m_0$, it results that $m > m_0 - 1$. The base polynomials for this operator are (for more details, please see [1]):

$$\varphi_{m,k}(x) = \frac{(m-1)x}{mx-1}\left(1-\frac{1}{m}\right)^{-m}\binom{m}{k}(1-x)^{m-k}\left(x-\frac{1}{m}\right)^k \tag{1.1}$$

If $m \in \mathbb{N}$, $m \geq m_0$, then the operators B_m^* are linear and positive.

Lemma 1.1. ([1]) The identities:

$$(B_m^* e_0)(x) = \frac{(m-1)x}{mx-1}$$
$$(B_m^* e_1)(x) = x$$
$$(B_m^* e_0)(x) = x^2$$

holds, for any $m \in \mathbb{N}$, $m \geq m_0$ and $x \in \left[\frac{1}{m_0-1}, 1\right]$.

More general, the King type operators, introduced by J.P.King in paper [2], have the following form:

$$V_n(f;x) = \sum_{k=0}^{m}\binom{m}{k}(r_m(x))^k(1-r_m(x))^{m-k} f\left(\frac{m}{k}\right) \tag{1.2}$$

where, $r_m : [0,1] \to [0,1]$, and

$$r_m(x) = \begin{cases} x^2, & \text{if } m = 1 \\ -\frac{1}{2(m-1)} + \sqrt{\frac{m}{m-1}x^2 + \frac{1}{4(m-1)^2}}, & \text{if } m = 2, 3, ... \end{cases} \tag{1.3}$$

Polynomials in Bernstein form were first used by Bernstein for the proof of the Stone - Weierstrass approximation theorem. With the advent of computer graphics, Bernstein polynomials, restricted to the interval $x \in [0,1]$, became important in the form of Bézier curves. The construction of a Bézier curve is based on the classical Bernstein polynomials $p_{m,k}(t) = \binom{m}{k}(1-t)^{m-k}t^k$, $k = 0, 1, 2, ..., m$ and have the following form:

$$B(t) = \sum_{k=0}^{m} \binom{m}{k} (1-t)^{m-k} t^k P_k,$$

where P_k, $k = 0, 1, 2, ..., m$ are the control points of the curve.
The cubic Bézier curve is obtained when we take $m = 3$, and has the following form:

$$B(t) = (1-t)^3 P_0 + 3(1-t)^2 t P_1 + 3(1-t)t^2 P_2 + t^3 P_3$$

The quadratic Bézier curve is obtained when we take $m = 2$, and has the following form:

$$B(t) = (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2$$

Definition 1.2. The convex hull of the set of points $X = \{x_0, x_1, ..., x_n\}$ is defined to be the set:

$$CH(X) = \left\{ a_0 x_0 + ... + a_n x_n | \sum_{i=1}^{n} a_i = 1, a_i \geq 0 \right\}$$

Some others important properties of the classical Bézier curves are:

- *Geometry Invariance Property*: Partition of unity property of the Bernstein polynomials, assures the invariance of the shape of the Bézier curves under translations and rotations of its control points.
- *End Points Geometric Property*: The first and the last control points are on the curve. The curve is tangent to the control polygon at the end points.
- *Variation Diminishing Property in 2 D*: The number of intersections of a straight line with a planar Bézier curve is no greater than the number of intersections of the line with the control polygon.

2. Main Result

Definition 2.1. A *K-Bézier* curve can be defined in the following way:

$$B_k(x) = \sum_{k=0}^{m} r_m(x) P_k; \quad m \geq 2$$

Here, P_k with $k = 0, 1, ..., m$ represent the control points of the curve and $r_m(x)$ are the King type polynomials introduced in (1.3).

Theorem 2.2. (convex hull property of the K-Bézier curves) Every point of the K-Bézier curve is in the interior of convex hull defined by the control point of the curve.

Proof. We know that the convex hull of a set of points X, can be write as a set

$$CH(X) = \left\{ a_0 x_0 + ... + a_n x_n | \sum_{i=1}^{n} a_i = 1, a_i \geq 0 \right\}.$$

For Bernstein polynomials, $b_{k,m}(t)$, a key property was (the partition of unity):

$$\sum_{k=0}^{m} b_{k,m}(t) = 1.$$

For ours K-Bézier curves, we have a King type operator, which generalized the classical Bernstein polynomials and also preserve this key property. So, we have

$$\sum_{k=0}^{m} r_m(t) = 1.$$

Using this and the Definition 1.2, one obtains the assertion of the theorem.

□

Theorem 2.3. The following inequality $B_{K_2}(t) \leq B_2(t)$ hold for $t \in [0,1]$ and for the same control points of the curves, where $B_{K_2}(t)$ represent the quadratic K-Bézier curve and $B_2(t)$ represent the classical quadratic Bézier curve.

Proof. The classical quadratic Bézier curve is:

$$B_2(t) = \sum_{k=0}^{2} \binom{2}{k} (1-t)^{2-k} t^k = (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2 \tag{2.1}$$

The quadratic K-Bézier curve obtained for $m = 2$, for the same control points, is:

$$B_{K_2}(t) = \left(\frac{3 - \sqrt{8t^2+1}}{2} \right)^2 P_0 + 2 \left(\frac{\sqrt{8t^2+1} - 1}{2} \right) \left(\frac{3 - \sqrt{8t^2+1}}{2} \right) P_1 + \left(\frac{\sqrt{8t^2+1} - 1}{2} \right)^2 P_2 \tag{2.2}$$

Using (2.1) and (2.2) and because $t \in [0,1]$, one obtains the assertion of the theorem.

□

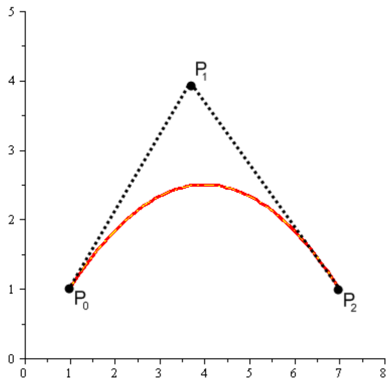
Example 2.4. Let us consider the following control points: $P_0(1,1); P_1(4,4)$ and $P_2(7,1)$. After easy computations, one obtains:
– For the classical Bézier curve, the following parametric equation:

$$\begin{cases} x(t) = 6t + 1 \\ y(t) = -6t^2 - 6t + 1 \end{cases}$$

– For the same control points, we get for the quadratic K-Bézier curve:

$$\begin{cases} x(t) = 1 + 6 \left(\frac{-1+\sqrt{8t^2+1}}{2} \right)^2 \\ y(t) = 1 + 6 \left(\frac{3-\sqrt{8t^2+1}}{2} \right) \left(\frac{-1+\sqrt{8t^2+1}}{2} \right) \end{cases}$$

Next, we plot the graphs for this quadratic Bézier type curves. The graph of classical quadratic Bézier curve is plotted with red and the graph of quadratic K-Bézier curve is plotted with yellow. In this graph we see how close this two curves are, for the same control points. This result confirm the assertion of theorem 2.3.



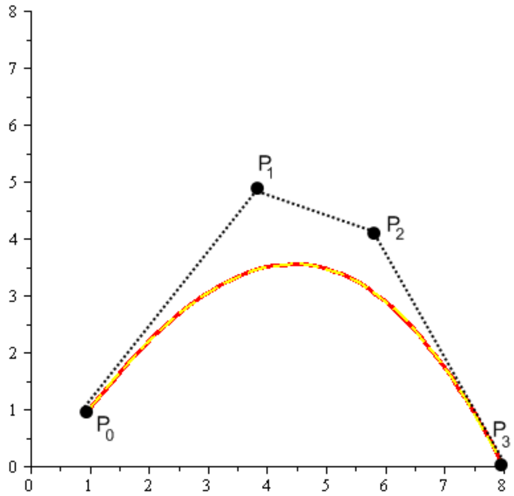
Example 2.5. Let us now, consider a Bézier cubic curve and K-Bézier cubic curve, generated by the following control points: $P_0(1,1); P_1(4,5); P_2(6,4)$ and $P_3(8,0)$.
After easy computations, one obtains:
– For the classical cubic Bézier curve, the following parametric equation:

$$\begin{cases} x(t) = t^3 - 3t^2 + 9t + 1 \\ y(t) = 2t^3 - 15t^2 + 12t + 1 \end{cases}$$

– For the same control points, we get for the cubic K-Bézier curve:

$$\begin{cases} x(t) = -\frac{27}{16} + \frac{43}{16}\sqrt{24t^2+1} - \frac{45}{8}t^2 + \frac{3}{8}t^2\sqrt{24t^2+1} \\ y(t) = -\frac{43}{4} + \frac{47}{4}\sqrt{24t^2+1} - \frac{297}{4}t^2 + \frac{21}{4}t^2\sqrt{24t^2+1} \end{cases}$$

Next, we plot the graphs for this cubic Bézier type curves. The graph of classical cubic Bézier curve is plotted with red and the graph of quadratic K-Bézier curve is plotted with yellow. Also, we can remark how close this two curves are, for the same control points.



Remark 2.6. For the King type operator (polynomials) obtained in paper [1], after easy computations, one obtains:

$$\sum_{k=0}^{m} \varphi_{m,k}(t) \neq 1$$

which means that we can not preserve the classical properties of Bézier curves especially the geometric invariance property. For the classical King operators (polynomials) described in (1.3) this important property of Bézier curves is preserved.

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