Acta Universitatis Apulensis No. 31/2012
ISSN: 1582-5329 pp. 321-325

ON THE CARNOT THEOREM IN THE POINCAR UPPER HALF-PLANE MODEL OF HYPERBOLIC GEOMETRY

Cătălin Barbu and Nilgün Sönmez

Abstract. In this study, we give a hyperbolic version of the Carnot theorem in the Poincaré upper half-plane model.

2000 *Mathematics Subject Classification*: 30F45, 51M10.

Keywords and phrases: hyperbolic geometry, hyperbolic triangle, Carnot's theorem, Poincaré upper half-plane model

1. Introduction

Hyperbolic geometry appeared in the first half of the 19^{th} century as an attempt to understand Euclid's axiomatic basis of geometry. It is also known as a type of non-euclidean geometry, being in many respects similar to euclidean geometry. Hyperbolic geometry includes similar concepts as distance and angle. Both these geometries have many results in common but many are different. Several useful models of hyperbolic geometry are studied in the literature as, for instance, the Poincaré disc and ball models, the Poincaré half-plane model, and the Beltrami-Klein disc and ball models [1] etc. Following [3] and [4] and earlier discoveries, the Beltrami-Klein model is also known as the Einstein relativistic velocity model. Here, in this study, we give a hyperbolic version of the Carnot theorem in the Poincaré upper half-plane model. The well-known the Carnot theorem states that if the points A', B', and C' be located on the sides BC, AC, and respectively AB of the triangle ABC, then the perpendiculars to the sides of the triangle at points A', B', and C' are concurrent if and only if

$$AC'^2 - BC'^2 + BA'^2 - CA'^2 + CB'^2 - AB'^2 = 0.1 \qquad (1)$$

The standard simple proof is based on the theorem of Pythagoras. For more details we refer to the monograph of C. Barbu [1], J. Gabay [3], L. Nicolescu, V. Boskoff [4]. We mention that O. Demirel and E. Soytürk [2] gave the hyperbolic form of Carnot's theorem in the Poincaré disc model of hyperbolic geometry. In order to introduce the Carnot's theorem into the Poincaré upper half-plane we refer briefly some facts about the Poincaré upper half-plane.

2. Preliminaries

The nature of the x-axis is such as to make impossible any communication between the lower and the upper half-planes. We restrict our attention to the upper half-plane and refer to it as the hyperbolic plane. It is also known as the Poincaré upper half-plane. The geodesic segments of the Poincaré upper half-plane (hyperbolic plane) are either segments of Euclidean straight lines that are perpendicular to the x-axis or arc of Euclidean semicircles that are centered on the x-axis. The hyperbolic length of the Euclidean line segment joining the points $P = (a; y_1)$ and $Q = (a; y_2)$, $0 < y_1 \leq y_2$, is $\ln \frac{y_2}{y_1}$.

The hyperbolic length between the points P and Q on a Euclidean semicircle with center $C = (c; 0)$ and radius r such that the radii CP and CQ make angles α and β ($\alpha < \beta$) respectively, with the positive x-axis [5],

$$\ln \frac{\csc \beta - \cot \beta}{\csc \alpha - \cot \alpha}.$$

Theorem 1. *Let ABC be a hyperbolic triangle with a right angle at C. If a, b, c, are the hyperbolic lengths of the sides opposite A, B, C, respectively, then*

$$\cosh c = \cosh a \cdot \cosh b.2 \tag{2}$$

For the proof of the theorem see [5].

3. The hyperbolic Carnot theorem in the Poincaré upper half-plane model of hyperbolic geometry

In this section, we prove Carnot's theorem in the Poincaré upper half-plane model of hyperbolic geometry.

Theorem 2. *Let $\triangle ABC$ be a hyperbolic triangle. Let the points A', B', and C' be located on the sides BC, CA and AB of the hyperbolic triangle ABC respectively. If the perpendiculars to the sides of the hyperbolic triangle at the points A', B', and C' are concurrent in the point M, then the following relations hold:*

$$i) \cosh MA'(\cosh A'B - \cosh A'C) + \cosh MB'(\cosh B'C - \cosh B'A)$$

$$+\cosh MC'(\cosh C'A - \cosh C'B) = 0,3 \tag{3}$$

$$ii)\frac{\cosh A'B}{\cosh A'C}\cdot\frac{\cosh B'C}{\cosh B'A}\cdot\frac{\cosh C'A}{\cosh C'B} = 1.4 \tag{4}$$

Proof. We assume that perpendiculars meet at a point of $\triangle ABC$ and let denote this point by M. The geodesic segments $AM, BM, CM, A'M, B'M$, and $C'M$ split the hyperbolic polygon into six right-angled hyperbolic triangles (see Figure 1).

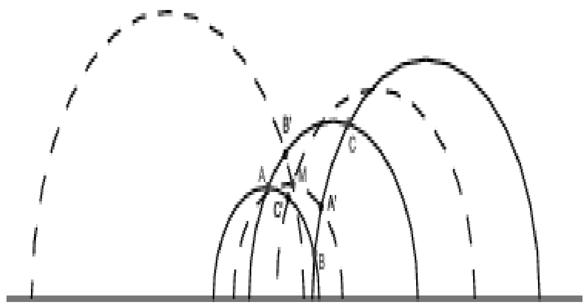

Figure 1

If we apply the Theorem 1 then

$$\cosh MA = \cosh MC'\cdot\cosh C'A = \cosh MB'\cdot\cosh B'A, 5 \tag{5}$$

$$\cosh MB = \cosh MA'\cdot\cosh A'B = \cosh MC'\cdot\cosh C'B, 6 \tag{6}$$

$$\cosh MC = \cosh MB'\cdot\cosh B'C = \cosh MA'\cdot\cosh A'C.7 \tag{7}$$

Adding the relations (5), (6) and (7) member by member, we get

$$\cosh MA'\cdot\cosh A'B + \cosh MB'\cdot\cosh B'C + \cosh MC'\cdot\cosh C'A =$$

$$\cosh MA'\cdot\cosh A'C + \cosh MB'\cdot\cosh B'A + \cosh MC'\cdot\cosh C'B,$$

and the equality (3) follows. Multiplying the relations (5), (6) and (7) member by member we obtain (4).

Naturally, one may wonder whether the converse of the Carnot theorem exists. Indeed, a partially converse theorem does exist as we show in the following theorem.

Theorem 3. *Let $\triangle ABC$ be a hyperbolic triangle. Let the points A', B', and C' be located on the sides BC, CA and AB of the hyperbolic triangle ABC respectively.*

If the perpendiculars to the sides of the hyperbolic triangle at the points B' and C' are concurrent in the point M and the following relation holds

$$\frac{\cosh A'B}{\cosh A'C} \cdot \frac{\cosh B'C}{\cosh B'A} \cdot \frac{\cosh C'A}{\cosh C'B} = 1,8 \tag{8}$$

then the point M is on the perpendicular to BC at the point A'.

Proof. Let A'' the feet of the perpendicular from M on the side BC. Using the already proven equality (4), we obtain

$$\frac{\cosh A''B}{\cosh A''C} \cdot \frac{\cosh B'C}{\cosh B'A} \cdot \frac{\cosh C'A}{\cosh C'B} = 19 \tag{9}$$

By (8) and (9) we get

$$\frac{\cosh A'B}{\cosh A'C} = \frac{\cosh A''B}{\cosh A''C}.10 \tag{10}$$

We note with x, y, z the hyperbolic distances BA', $A'A''$, and $A''C$ respectively. Then (10) is equivalent with

$$\cosh(x+y) \cdot \cosh(y+z) - \cosh x \cdot \cosh z = 011 \tag{11}$$

If we use the formula $\cosh\alpha \cdot \cosh\beta = \frac{\sinh(\alpha+\beta)-\sinh(\alpha-\beta)}{2}$, then the relation (11) is equivalent with

$$\frac{\sinh(x+2y+z) - \sinh(x-z)}{2} - \frac{\sinh(x+z) - \sinh(x-z)}{2} = 0$$

or

$$\sinh(x+2y+z) = \sinh(x+z).12 \tag{12}$$

By (12) using injectivity of the function $\sinh x$ we get $y = 0$. Then the points A' and A'' are identical.

References

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Cătălin Barbu
Mathematics Department
Vasile Alecsandri National College
Bacău, Romania
email: *kafka_mate@yahoo.com*

Nilgün Sönmez
Mathematics Department
Faculty of Science and Literatures
Afyon Kocatepe University
Ans Campus, 03200-Afyonkarahisar, Turkey
email: *nceylan@aku.edu.tr*