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A GEOMETRIC WAY TO GENERATE BLUNDON TYPE INEQUALITIES

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Abstract. We present a geometric way to generate Blundon type inequalities. Theorem 3.1 gives the formula for $\cos\widehat{POQ}$ in terms of the barycentric coordinates of the points P and Q with respect to a given triangle. This formula implies Blundon type inequalities generated by the points P and Q . Some applications are given in the last section by choosing special points P and Q.

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1. Introduction

Consider O the circumcenter, I the incenter , G the centroid, N the Nagel point, s the semiperimeter, R the circumradius, and r the inradius of triangle ABC.

Blundon's inequalities express the necessary and sufficient conditions for the existence of a triangle with elements s, R and r:

$$2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr} \leq s^2 \leq 2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}. \tag{1}$$

Clearly these two inequalities can be written in the following equivalent form

$$|s^2-2R^2-10Rr+r^2| \leq 2(R-2r)\sqrt{R^2-2Rr}, \tag{2}$$

and in many references this relation is called the fundamental inequality of triangle ABC.

The standard proof is an algebraic one, it was first time given by W.J.Blundon [5] and it is based on the characterization of cubic equations with the roots the length sides of a triangle. For more details we refer to the monograph of D. Mitrinović, J. Pečarić, V. Volenec [16], and to the papers of C.Niculescu [17],[18]. R.A.Satnoianu [20], and S.Wu [22] have obtained some improvements of this important inequality.

The following result was obtained by D.Andrica and C.Barbu in the paper [3] and it contains a simple geometric proof of (1). Assume that the triangle ABC is not equilateral. The following relation holds :

$$\cos\widehat{ION} = \frac{2R^2 + 10Rr - r^2 - s^2}{2(R-2r)\sqrt{R^2 - 2Rr}}. \tag{3}$$

If we have $R = 2r$, then the triangle must be equilateral and we have equality in (1) and (2). If we assume that $R - 2r \neq 0$, then inequalities (1) are direct consequences of the fact that $-1 \leq \cos\widehat{ION} \leq 1$.

In this geometric argument the main idea is to consider the points O, I and N, and then to get the formula (3). It is a natural question to see what is a similar formula when we kip the circumcenter O and we replace the points I and N by other two points P and Q. In this way we obtain Blundon type inequalities generated by the points P and Q. Section 2 contains the basic facts about the main ingredient helping us to do all the computations, that is the barycentric coordinates. In Section 3 we present the analogous formula to (3), for the triangle POQ, and the we derive the Blundon type inequalities generated in this way. The last section contains some applications of the results in Section 3 as follows: the classical Blundon's inequalities, the dual Blundon's inequalities obtained in the paper [3], the Blundon's inequalities generated by two Cevian points of rank $(k; l; m)$.

2. Some basic results about barycentric coordinates

Let P be a point situated in the plane of the triangle ABC. The Cevian triangle DEF is defined by the intersection of the Cevian lines though the point P and the sides BC, CA, AB of triangle. If the point P has barycentric coordinates $t_1 : t_2 : t_3$, then the vertices of the Cevian triangle DEF have barycentric coordinates given by: $D(0 : t_2 : t_3), E(t_1 : 0 : t_3)$ and $F(t_1 : t_2 : 0)$. The barycentric coordinates were introduced in 1827 by Möbius (see [10]). The using of barycentric coordinates defines a distinct part of Geometry called Barycentric Geometry. More details can be found in the monographs of C. Bradley [10], C. Coandă[11], C. Coşniţă [12], C. Kimberling [14], and in the papers of O. Bottema [9], J. Scott [21], and P. Yiu [23].

It is well-known ([11],[12]) that for every point M in the plane of triangle ABC, then the following relation holds:

$$(t_1+t_2+t_3)\overrightarrow{MP} = t_1\overrightarrow{MA} + t_2\overrightarrow{MB} + t_3\overrightarrow{MC}. \tag{4}$$

In the particular case when $M \equiv P$, we obtain

$$t_1\overrightarrow{PA} + t_2\overrightarrow{PB} + t_3\overrightarrow{PC} = \overrightarrow{0}.$$

This last relation shows that the point P is the barycenter of the system $\{A, B, C\}$ with the weights $\{t_1, t_2, t_3\}$. The following well-known result is very useful in computing distances from the point M to the barycenter P of the system $\{A, B, C\}$ with the weights $\{t_1, t_2, t_3\}$.

Theorem 2.1. *If M is a point situated in the plane of triangle ABC, then*

$$(t_1+t_2+t_3)^2MP^2 = (t_1MA^2+t_2MB^2+t_3MC^2)(t_1+t_2+t_3) - (t_2t_3a^2+t_3t_1b^2+t_1t_2c^2). \tag{5}$$

Proof. Using the scalar product of two vectors, from (4) we obtain:

$$(t_1+t_2+t_3)^2MP^2 = t_1^2MA^2 + t_2^2MB^2 + t_3^2MC^2 +$$

$$2t_1t_2\overrightarrow{MA}\cdot\overrightarrow{MB} + 2t_1t_3\overrightarrow{MA}\cdot\overrightarrow{MC} + 2t_2t_3\overrightarrow{MB}\cdot\overrightarrow{MC},$$

that is

$$(t_1+t_2+t_3)^2MP^2 = t_1^2MA^2 + t_2^2MB^2 + t_3^2MC^2 +$$

$$t_1t_2(MA^2+MB^2-AB^2) + t_1t_3(MA^2+MC^2-AC^2) + t_2t_3(MB^2+MC^2-BC^2),$$

hence,

$$(t_1+t_2+t_3)^2MP^2 = (t_1MA^2+t_2MB^2+t_3MC^2)(t_1+t_2+t_3) - (t_2t_3a^2+t_3t_1b^2+t_1t_2c^2).$$

To get the last relation we have used the definition of the scalar product and the Cosine Law as follows

$$2\overrightarrow{MA}\cdot\overrightarrow{MB} = 2MA\cdot MB\cos\widehat{AMB} =$$

$$2MA\cdot MB\cdot\frac{MA^2+MB^2-AB^2}{2MA\cdot MB} = MA^2+MB^2-AB^2.$$

If we consider that t_1, t_2, t_3, and $t_1+t_2+t_3$ are nonzero real numbers, then the relation (5) becomes the Lagrange's relation

$$MP^2 = \frac{t_1MA^2+t_2MB^2+t_3MC^2}{t_1+t_2+t_3} - \frac{t_1t_2t_3}{(t_1+t_2+t_3)^2}\left(\frac{a^2}{t_1}+\frac{b^2}{t_2}+\frac{c^2}{t_3}\right). \tag{6}$$

If we consider in (6) $M \equiv O$, the circumcenter of the triangle, then it follows

$$R^2 - OP^2 = \frac{t_1 t_2 t_3}{(t_1+t_2+t_3)^2}\left(\frac{a^2}{t_1} + \frac{b^2}{t_2} + \frac{c^2}{t_3}\right). \tag{7}$$

The following version of Cauchy-Schwarz inequality is also known in the literature as Bergström's inequality (see [6], [7], [8], [19]): If $x_k, a_k \in \mathbb{R}$ and $a_k > 0$, $k = 1, 2, ..., n$, then

$$\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + ... + \frac{x_n^2}{a_n} \geq \frac{(x_1 + x_2 + ... + x_n)^2}{a_1 + a_2 + ... + a_n},$$

with equality if and only if

$$\frac{x_1}{a_1} = \frac{x_2}{a_2} = ... = \frac{x_n}{a_n}.$$

Using Bergström's inequality and relation (4), we obtain

$$R^2 - OP^2 \geq \frac{t_1 t_2 t_3}{(t_1+t_2+t_3)^2} \cdot \frac{(a+b+c)^2}{t_1+t_2+t_3},$$

that is in any triangle with semiperimeter s the following inequality holds:

$$R^2 - OP^2 \geq \frac{4s^2 t_1 t_2 t_3}{(t_1+t_2+t_3)^3},$$

where $t_1 : t_2 : t_3$ are the barycentric coordinates of P and $t_1, t_2, t_3 > 0$. Equality holds if an only if $t_1 = a, t_2 = b, t_3 = c$, that is $P \equiv I$, the incenter of the triangle ABC.

Theorem 2.2. ([11], [12]). *If the points P and Q have barycentric coordinates $t_1 : t_2 : t_3$, and $u_1 : u_2 : u_3$, respectively, with respect to the triangle ABC, and $u = u_1 + u_2 + u_3$, $t = t_1 + t_2 + t_3$, then*

$$PQ^2 = -\alpha\beta\gamma\left(\frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma}\right) \tag{8}$$

where the numbers α, β, γ are defined by

$$\alpha = \frac{u_1}{u} - \frac{t_1}{t}; \beta = \frac{u_2}{u} - \frac{t_2}{t}; \gamma = \frac{u_3}{u} - \frac{t_3}{t}.$$

3. Blundon type inequalities generated by two points

Theorem 3.1. *Let P and Q be two points diferent from the circumcircle O, having the barycentric coordinates $t_1 : t_2 : t_3$, and $u_1 : u_2 : u_3$ with respect to the triangle ABC and let $u = u_1 + u_2 + u_3$, $t = t_1 + t_2 + t_3$. If $u_1, u_2, u_3, t_1, t_2, t_3 \neq 0$, then the following relation holds*

$$\cos\widehat{POQ}=\frac{2R^2 - \frac{t_1t_2t_3}{t^2}\left(\frac{a^2}{t_1}+\frac{b^2}{t_2}+\frac{c^2}{t_3}\right) - \frac{u_1u_2u_3}{t^2}\left(\frac{a^2}{u_1}+\frac{b^2}{u_2}+\frac{c^2}{u_3}\right) + \alpha\beta\gamma\left(\frac{a^2}{\alpha}+\frac{b^2}{\beta}+\frac{c^2}{\gamma}\right)}{2\sqrt{\left[R^2 - \frac{t_1t_2t_3}{t^2}\left(\frac{a^2}{t_1}+\frac{b^2}{t_2}+\frac{c^2}{t_3}\right)\right]\cdot\left[R^2 - \frac{u_1u_2u_3}{u^2}\left(\frac{a^2}{u_1}+\frac{b^2}{u_2}+\frac{c^2}{u_3}\right)\right]}} \tag{9}$$

where a, b, c are the length sides of the triangle and

$$\alpha = \frac{u_1}{u} - \frac{t_1}{t}; \beta = \frac{u_2}{u} - \frac{t_2}{t}; \gamma = \frac{u_3}{u} - \frac{t_3}{t}. \tag{10}$$

Proof. Applying the relation (7) for the points P and Q, we have

$$OP^2 = R^2 - \frac{t_1t_2t_3}{t^2}\left(\frac{a^2}{t_1}+\frac{b^2}{t_2}+\frac{c^2}{t_3}\right) \tag{11}$$

and

$$OQ^2 = R^2 - \frac{u_1u_2u_3}{u^2}\left(\frac{a^2}{u_1}+\frac{b^2}{u_2}+\frac{c^2}{u_3}\right) \tag{12}$$

We use the Law of Cosines in the triangle POQ to obtain

$$\cos\widehat{POQ} = \frac{OP^2 + OQ^2 - PQ^2}{2OP \cdot OQ}, \tag{13}$$

and from relations (8), (11), (12) and (13) we obtain the relation (9).

Theorem 3.2. *Let P and Q be two points diferent from the circumcircle O, having the barycentric coordinates $t_1 : t_2 : t_3$, and $u_1 : u_2 : u_3$ with respect to the triangle ABC and let $u = u_1 + u_2 + u_3$, $t = t_1 + t_2 + t_3$. If $u_1, u_2, u_3, t_1, t_2, t_3 \neq 0$, then the following relation holds*

$$-2\sqrt{\left[R^2 - \frac{t_1t_2t_3}{t^2}\left(\frac{a^2}{t_1} + \frac{b^2}{t_2} + \frac{c^2}{t_3}\right)\right] \cdot \left[R^2 - \frac{u_1u_2u_3}{u^2}\left(\frac{a^2}{u_1} + \frac{b^2}{u_2} + \frac{c^2}{u_3}\right)\right]} \leq$$

$$\alpha\beta\gamma\left(\frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma}\right) + 2R^2 - \left[\frac{t_1t_2t_3}{t^2}\left(\frac{a^2}{t_1} + \frac{b^2}{t_2} + \frac{c^2}{t_3}\right) + \frac{u_1u_2u_3}{u^2}\left(\frac{a^2}{u_1} + \frac{b^2}{u_2} + \frac{c^2}{u_3}\right)\right] \leq$$

$$2\sqrt{\left[R^2 - \frac{t_1t_2t_3}{t^2}\left(\frac{a^2}{t_1} + \frac{b^2}{t_2} + \frac{c^2}{t_3}\right)\right] \cdot \left[R^2 - \frac{u_1u_2u_3}{u^2}\left(\frac{a^2}{u_1} + \frac{b^2}{u_2} + \frac{c^2}{u_3}\right)\right]} \qquad (14)$$

where a, b, c are the length sides of the triangle and the numbers α, β, γ are defined by (10).

Proof. The inequalities (14) are simple direct consequences of the fact that $-1 \leq \cos\widehat{POQ} \leq 1$. The equality in the right inequality holds if and only if $\widehat{POQ} = 0$, that is the points O, P, Q are collinear in the order O, P, Q or O, Q, P. The equality in the left inequality holds if and only if $\widehat{POQ} = \pi$, that is the points O, P, Q are collinear in the order P, O, Q or Q, O, P.

From Theorem 3.1. it follows that it is a natural and important problem to construct the triangle ABC from the points O, P, Q, when we know their barycentric coordinates. In the special case when $P \equiv I$ and $Q \equiv N$ we know that that points I, G, N are collinear, determining the Nagel line of triangle, and the centroid G lies on the segment IN such that $IG = \frac{1}{3}IN$. Then, using the Euler's line of the triangle, we get the orthocenter H on the ray $(OG$ such that $OH = 3OG$. In this case the problem is reduced to the famous Euler's determination problem i.e. to construct a triangle from its incenter I, circumcenter O, and orthocenter H (see the paper of P.Yiu [24] for details and results). This is a reason to call the problem as the general determination problem.

4. Applications

The formula (3) and the classical Blundon's inequalities (1) can be obtained from (9) and (14) by considering $P = I$, the incenter, and $Q = N$, the Nagel point of the triangle. Indeed, the barycentric coordinates of incenter I and of Nagel's point N are $(t_1, t_2, t_3) = (a, b, c)$, and $(u_1, u_2, u_3) = (s - a, s - b, s - c)$, respectively. We have

$$u = u_1 + u_2 + u_3 = s, \quad u_1u_2u_3 = r^2 s, \qquad (15)$$

and

$$t = t_1 + t_2 + t_3 = 2s,\ t_1t_2t_3 = abc = 4Rrs. \tag{16}$$

We obtain

$$\alpha = \frac{s-a}{s} - \frac{a}{2s} = \frac{2s-3a}{2s},\ \beta = \frac{2s-3b}{2s},\ \gamma = \frac{2s-3c}{2s}. \tag{17}$$

Therefore

$$\alpha\beta\gamma\left(\frac{a^2}{\alpha}+\frac{b^2}{\beta}+\frac{c^2}{\gamma}\right) = \sum_{cyc}\beta\gamma a^2 = \sum_{cyc}\left(1-\frac{3b}{2s}\right)\left(1-\frac{3c}{2s}\right)a^2 =$$

$$\sum_{cyc}a^2 - \frac{3}{2s}\sum_{cyc}\left[a^2(a+b+c)-a^3\right] + \frac{9abc}{4s^2}\sum_{cyc}a =$$

$$\sum_{cyc}a^2 - 3\sum_{cyc}a^2 + \frac{3}{2s}\sum_{cyc}a^3 + \frac{9abc}{2s} =$$

$$-2(2s^2-2r^2-8Rr) + 3(s^2-3r^2-6Rr) + 18Rr$$

that is

$$\alpha\beta\gamma\left(\frac{a^2}{\alpha}+\frac{b^2}{\beta}+\frac{c^2}{\gamma}\right) = -s^2-5r^2+16Rr. \tag{18}$$

Now, using (16) and (17) we get

$$\frac{t_1t_2t_3}{t}\left(\frac{a^2}{t_1}+\frac{b^2}{t_2}+\frac{c^2}{t_3}\right) = \frac{4Rrs}{4s^2}\cdot 2s = 2Rr, \tag{19}$$

and

$$\frac{u_1u_2u_3}{u}\left(\frac{a^2}{u_1}+\frac{b^2}{u_2}+\frac{c^2}{u_3}\right) = \frac{r^2s}{s^2}\left(\frac{a^2}{s-a}+\frac{b^2}{s-b}+\frac{c^2}{s-c}\right) =$$

$$\frac{r^2}{s}\cdot\frac{\sum\limits_{cyc}a^2(s-b)(s-c)}{r^2s} =$$

$$\frac{1}{s^2}\left\{s^2\sum_{cyc}a^2 - s\sum_{cyc}\left[a^2(a+b+c)-a^3\right] + abc\sum_{cyc}a\right\} =$$

$$\frac{1}{s^2}\left(s\sum_{cyc}a^3 - s^2\sum_{cyc}a^2 + 8Rrs^2\right) =$$

$$\frac{1}{s^2}\left[2s^2(s^2-3r^2-6Rr) - s^2(2s^2-2r^2-8Rr) + 8Rrs^2\right] = 4Rr - 4r^2 \tag{20}$$

Using the relations (18)-(20) in (9) we obtain the relation (3). These computations are similar to those given by complex numbers in [1].

Now, consider the excenters I_a, I_b, I_c, and N_a, N_b, N_c the adjoint points to the Nagel point N. For the definition and some properties of the adjoint points N_a, N_b, N_c we refer to the paper of D.Andrica and K.L.Nguyen [2]. Let s, R, r, r_a, r_b, r_c be the semiperimeter, circumradius, inradius, and exradii of triangle ABC, respectively. Considering the triangle I_aON_a, D.Andrica and C.Barbu [3] have proved the following formula

$$\cos \widehat{I_aON_a} = \frac{R^2 - 3Rr_a - r_a^2 - \alpha}{(R+2r_a)\sqrt{R^2+2Rr_a}}, \tag{21}$$

where $\alpha = \frac{a^2+b^2+c^2}{4}$.

Using formula (21),we get the dual form of Blundon's inequalities given in the paper [3]

$$0 \leq \frac{a^2+b^2+c^2}{4} \leq R^2 - 3Rr_a - r_a^2 + (R+2r_a)\sqrt{R^2+2Rr_a}. \tag{22}$$

There are similar inequalities involving the exradii r_b and r_c.

We known that the barycentric coordinates of the excenter I_a are $(t_1, t_2, t_3) = (-a, b, c)$, and of the adjoint Nagel point N_a are $(u_1, u_2, u_3) = (s, c-s, b-s)$. Using formula (9) we can obtain the relation (21) and then the dual form of the classical Blundon's inequalities (22).

We have

$$u = u_1 + u_2 + u_3 = s - a, \ u_1u_2u_3 = s(s-b)(s-c)$$

and

$$t = t_1 + t_2 + t_3 = 2(s-a), \ t_1t_2t_3 = -abc = -4Rrs.$$

We obtain

$$\alpha = \frac{2s+a}{2(s-a)} = 1 + \frac{3a}{2(s-a)},$$

$$\beta = \frac{2c-2s-b}{2(s-a)} = 1 - \frac{3b}{2(s-a)},$$

$$\gamma = \frac{2b-2s-c}{2(s-a)} = 1 - \frac{3c}{2(s-a)}.$$

Therefore,

$$\frac{t_1t_2t_3}{t^2}\left(\frac{a^2}{t_1} + \frac{b^2}{t_2} + \frac{c^2}{t_3}\right) = \frac{-4Rrs}{4(s-a)^2} \cdot 2(s-a) = -2Rr_a, \tag{23}$$

and

$$\frac{u_1u_2u_3}{u^2}\left(\frac{a^2}{u_1}+\frac{b^2}{u_2}+\frac{c^2}{u_3}\right)=\frac{s(s-c)(s-b)}{(s-a)^2}\left(\frac{a^2}{s}-\frac{b^2}{s-c}-\frac{c^2}{s-b}\right)=$$

$$-\left(-a^2\cdot\frac{s-b}{s-a}\cdot\frac{s-c}{s-a}+b^2\cdot\frac{s}{s-a}\cdot\frac{s-b}{s-a}+c^2\cdot\frac{s}{s-a}\cdot\frac{s-c}{s-a}\right)=$$

$$-\left(-a^2\cdot\frac{r_a}{r_b}\cdot\frac{r_a}{r_c}+b^2\cdot\frac{r_a}{r}\cdot\frac{r_a}{r_b}+c^2\cdot\frac{r_a}{r}\cdot\frac{r_a}{r_c}\right)=$$

$$-r_a^2\left(\frac{-a^2}{r_br_c}+\frac{b^2}{rr_b}+\frac{c^2}{rr_c}\right)=-r_a^2\left(\frac{4R}{r_a}+4\right)=-4Rr_a-4r_a^2, \tag{24}$$

where we have used the relation $\frac{-a^2}{r_br_c}+\frac{b^2}{rr_b}+\frac{c^2}{rr_c}=\frac{4R}{r_a}+4$ (see [2], p. 134).

Now, we will calculate the expression:

$$E=\alpha\beta\gamma\left(\frac{a^2}{\alpha}+\frac{b^2}{\beta}+\frac{c^2}{\gamma}\right)+\frac{a^2+b^2+c^2}{2}=$$

$$a^2\beta\gamma+\frac{a^2}{2}+b^2\alpha\gamma+\frac{b^2}{2}+c^2\alpha\beta+\frac{c^2}{2}=$$

$$a^2\left[1-\frac{3(b+c)}{2(s-a)}+\frac{9bc}{4(s-a)^2}\right]+\frac{a^2}{2}+$$

$$b^2\left[1+\frac{3(a-c)}{2(s-a)}-\frac{9ca}{4(s-a)^2}\right]+\frac{b^2}{2}+$$

$$c^2\left[1+\frac{3(a-b)}{2(s-a)}-\frac{9ab}{4(s-a)^2}\right]+\frac{c^2}{2},$$

that is

$$E=a^2\left[\frac{-3s}{2(s-a)}+\frac{9bc}{4(s-a)^2}\right]+b^2\left[\frac{3(s-c)}{2(s-a)}-\frac{9ca}{4(s-a)^2}\right]+$$

$$c^2\left[\frac{3(s-b)}{2(s-a)}-\frac{9ab}{4(s-a)^2}\right]=$$

$$\frac{3}{2(s-a)}[-a^2s+b^2(s-c)+c^2(s-b)]+\frac{9abc}{4(s-a)^2}(a-b-c)=$$

$$\frac{3}{2(s-a)}[s(-a^2+b^2+c^2)-bc(b+c)]-18Rr_a. \tag{25}$$

We have

$$s(-a^2+b^2+c^2)-bc(b+c)=2sbc\cos A-2bcs+abc=$$

$$2sbc(\cos A - 1) + abc = abc - 4sbc\sin^2\frac{A}{2} =$$
$$abc - 4s(p-b)(p-c) = abc - 4Sr_a = 4S(R - r_a), \tag{26}$$

where S denotes the area of triangle ABC. From relations (25) and (26) we get

$$E = \frac{3}{2(s-a)}\cdot 4S(R-r_a) - 18Rr_a = 6r_a(R-r_a) - 18Rr_a = -12Rr_a - 6r_a^2,$$

therefore

$$\alpha\beta\gamma\left(\frac{a^2}{\alpha}+\frac{b^2}{\beta}+\frac{c^2}{\gamma}\right) = -12Rr_a - 6r_a^2 - \frac{a^2+b^2+c^2}{2} \tag{27}$$

Using formulas (23), (24) and (27) in the general formula (9) we obtain the relation (21).

In the paper [13], N. Minculete and C. Barbu have introduced the Cevians of rank $(k;l;m)$. The line AD is called *ex-Cevian of rank* $(k;l;m)$ or *exterior Cevian of rank* $(k;l;m)$, if the point D is situated on side (BC) of the non-isosceles triangle ABC and the following relation holds:

$$\frac{BD}{DC} = \left(\frac{c}{b}\right)^k\cdot\left(\frac{s-c}{s-b}\right)^l\cdot\left(\frac{a+b}{a+c}\right)^m.$$

In the paper [13] it is proved that the Cevians of rank $(k;l;m)$ are concurrent in the point $I(k,l,m)$ called *the Cevian point of rank* $(k;l;m)$ and the barycentric coordinates of $I(k,l,m)$ are:

$$a^k(s-a)^l(b+c)^m : b^k(s-b)^l(a+c)^m : c^k(s-c)^l(a+b)^m.$$

In the case $l = m = 0$, we obtain the Cevian point of rank k.

Let I_1, I_2 be two Cevian points with barycentric coordinates:

$$I_i[a^{k_i}(s-a)^{l_i}(b+c)^{m_i} : b^{k_i}(s-b)^{l_i}(a+c)^{m_i} : c^{k_i}(s-c)^{l_i}(a+b)^{m_i}],\ i = 1,2.$$

Denote $t_i^1 = a^{k_i}(s-a)^{l_i}(b+c)^{m_i}$,$t_i^2 = b^{k_i}(s-b)^{l_i}(a+c)^{m_i}$, $t_i^3 = c^{k_i}(s-c)^{l_i}(a+b)^{m_i}$, $i = 1,2$. From formula (9) we obtain

$$\cos\widehat{I_1OI_2} = \frac{2R^2 - \frac{t_1^1t_1^2t_1^3}{(T_1)^2}\left(\frac{a^2}{t_1^1}+\frac{b^2}{t_1^2}+\frac{c^2}{t_1^3}\right) - \frac{t_2^1t_2^2t_2^3}{(T_2)^2}\left(\frac{a^2}{t_2^1}+\frac{b^2}{t_2^2}+\frac{c^2}{t_2^3}\right) + \alpha\beta\gamma\left(\frac{a^2}{\alpha}+\frac{b^2}{\beta}+\frac{c^2}{\gamma}\right)}{2\sqrt{\left[R^2 - \frac{t_1^1t_1^2t_1^3}{(T_1)^2}\left(\frac{a^2}{t_1^1}+\frac{b^2}{t_1^2}+\frac{c^2}{t_1^3}\right)\right]\cdot\left[R^2 - \frac{t_2^1t_2^2t_2^3}{(T_2)^2}\left(\frac{a^2}{t_2^1}+\frac{b^2}{t_2^2}+\frac{c^2}{t_2^3}\right)\right]}}, \tag{28}$$

where $T_1 = t_1^1 + t_1^2 + t_1^3, T_2 = t_2^1 + t_2^2 + t_2^3$, and for $i = 1, 2$, we have

$$\frac{t_i^1 t_i^2 t_i^3}{(T_i)^2}\left(\frac{a^2}{t_i^1}+\frac{b^2}{t_i^2}+\frac{c^2}{t_i^3}\right)=\frac{\prod\limits_{cyc} a^{k_i}(s-a)^{l_i}(b+c)^{m_i}}{\sum\limits_{cyc} a^{k_i}(s-a)^{l_i}(b+c)^{m_i}}\cdot\sum_{cyc}\frac{a^2}{a^{k_i}(s-a)^{l_i}(b+c)^{m_i}},$$

and

$$\alpha=\frac{a^{k_1}(s-a)^{l_1}(b+c)^{m_1}}{\sum\limits_{cyc} a^{k_1}(s-a)^{l_1}(b+c)^{m_1}}-\frac{a^{k_2}(s-a)^{l_2}(b+c)^{m_2}}{\sum\limits_{cyc} a^{k_2}(s-a)^{l_2}(b+c)^{m_2}},$$

$$\beta=\frac{b^{k_1}(s-b)^{l_1}(a+c)^{m_1}}{\sum\limits_{cyc} a^{k_1}(s-a)^{l_1}(b+c)^{m_1}}-\frac{b^{k_2}(s-b)^{l_2}(a+c)^{m_2}}{\sum\limits_{cyc} a^{k_2}(s-a)^{l_2}(b+c)^{m_2}},$$

$$\gamma=\frac{c^{k_1}(s-c)^{l_1}(a+b)^{m_1}}{\sum\limits_{cyc} a^{k_1}(s-a)^{l_1}(b+c)^{m_1}}-\frac{c^{k_2}(s-c)^{l_2}(a+b)^{m_2}}{\sum\limits_{cyc} a^{k_2}(s-a)^{l_2}(b+c)^{m_2}}.$$

If I_1, I_2 are Cevian points of rank k_1, k_2, then formula (28) becomes

$$\cos\widehat{I_1OI_2}=\frac{2R^2-(abc)^{k_1}\frac{S_{2-k_1}}{(S_{k_1})^2}-(abc)^{k_2}\frac{S_{2-k_2}}{(S_{k_2})^2}+\sum\limits_{cyc}(\frac{b^{k_1}}{S_{k_1}}-\frac{b^{k_2}}{S_{k_2}})(\frac{c^{k_1}}{S_{k_1}}-\frac{c^{k_2}}{S_{k_2}})a^2}{2\sqrt{[R^2-(abc)^{k_1}\frac{S_{2-k_1}}{(S_{k_1})^2}][R^2-(abc)^{k_2}\frac{S_{2-k_2}}{(S_{k_2})^2}]}}, \quad (29)$$

where $S_l = a^l + b^l + c^l$.

Here are few special cases of formula (29). For $k_1 = 0$ and $k_2 = 1$ we get the centroid G and the incenter I of barycentric coordinates $(1; 1; 1)$ and $(a; b; c)$, respectively. Formula (29) becomes

$$\cos\widehat{GOI}=\frac{6R^2-s^2-r^2+2Rr}{2\sqrt{9R^2-2s^2+2r^2+8Rr}\cdot\sqrt{R^2-2Rr}}, \quad (30)$$

where $abc = 4sRr$, $S_0 = 3$, $S_1 = 2s$, $S_2 = 2(s^2 - r^2 - 4Rr)$.

For $k_2 = 2$ we obtain the Lemoine point L of triangle ABC, of barycentric coordinates $(a^2; b^2; c^2)$, and other two formulas are generated

$$\cos\widehat{GOL}=\frac{6R^2S_2-S_2^2+S_4}{2\sqrt{9R^2-S_2}\cdot\sqrt{R^2S_2^2-48(Rrs)^2}}, \quad (31)$$

where $S_4 = S_2^2 - 2[(s^2 + r^2 + 4Rr)^2 - 16Rrs^2]$, and

$$\cos\widehat{IOL}=\frac{RS_2+rS_2-4rs^2}{2\sqrt{R^2-2Rr}\cdot\sqrt{S_2^2-48r^2s^2}}. \quad (32)$$

Each of the formulas (30), (31), (32) generates a Blundon type inequality, but these inequalities have not nice geometric interpretations.

Let I_1, I_2, I_3 be three Cevian points of rank $(k; l; m)$ with barycentric coordinates as follows:

$$a^{k_i}(s-a)^{l_i}(b+c)^{m_i} : b^{k_i}(s-b)^{l_i}(a+c)^{m_i} : c^{k_i}(s-c)^{l_i}(a+b)^{m_i},\ i = 1, 2, 3,$$

and let $t_i^1 = a^{k_i}(s-a)^{l_i}(b+c)^{m_i}$, $t_i^2 = b^{k_i}(s-b)^{l_i}(a+c)^{m_i}$, $t_i^3 = c^{k_i}(s-c)^{l_i}(a+b)^{m_i}$. Now, consider the numbers

$$\alpha_{ij} = \frac{t_j^1}{t_j^1 + t_j^2 + t_j^3} - \frac{t_i^1}{t_i^1 + t_i^2 + t_i^3},$$

and

$$\beta_{ij} = \frac{t_j^2}{t_j^1 + t_j^2 + t_j^3} - \frac{t_i^2}{t_i^1 + t_i^2 + t_i^3},$$

and

$$\gamma_{ij} = \frac{t_j^3}{t_j^1 + t_j^2 + t_j^3} - \frac{t_i^3}{t_i^1 + t_i^2 + t_i^3},$$

for all $i, j \in \{1, 2, 3\}$. Applying the relation (5) we obtain

$$I_i I_j^2 = -\alpha_{ij} \cdot \beta_{ij} \cdot \gamma_{ij} \cdot \left(\frac{a^2}{\alpha_{ij}} + \frac{b^2}{\beta_{ij}} + \frac{c^2}{\gamma_{ij}} \right),$$

for all $i, j \in \{1, 2, 3\}$. Using the Cosine Law in triangle $I_1I_2I_3$ it follows

$$\cos \widehat{I_1I_2I_3} = \frac{I_1I_2^2 + I_2I_3^2 - I_3I_1^2}{2I_1I_2 \cdot I_2I_3} =$$

$$\frac{-a^2(\beta_{12}\gamma_{12} + \beta_{23}\gamma_{23} - \beta_{31}\gamma_{31}) - b^2(\gamma_{12}\alpha_{12} + \gamma_{23}\alpha_{23} - \gamma_{31}\alpha_{31}) + c^2(\alpha_{12}\beta_{12} + \alpha_{23}\beta_{23} - \alpha_{31}\beta_{31})}{2\sqrt{-\beta_{12}\gamma_{12}a^2 - \gamma_{12}\alpha_{12}b^2 - \alpha_{12}\beta_{12}c^2} \cdot \sqrt{-\beta_{23}\gamma_{23}a^2 - \gamma_{23}\alpha_{23}b^2 - \alpha_{23}\beta_{23}c^2}} \tag{33}$$

Theorem 4.1. *The following inequalities hold*

$$-2\sqrt{-\beta_{12}\gamma_{12}a^2 - \gamma_{12}\alpha_{12}b^2 - \alpha_{12}\beta_{12}c^2} \cdot \sqrt{-\beta_{23}\gamma_{23}a^2 - \gamma_{23}\alpha_{23}b^2 - \alpha_{23}\beta_{23}c^2} \le$$

$$-a^2(\beta_{12}\gamma_{12}+\beta_{23}\gamma_{23}-\beta_{31}\gamma_{31}) - b^2(\gamma_{12}\alpha_{12}+\gamma_{23}\alpha_{23}-\gamma_{31}\alpha_{31}) + c^2(\alpha_{12}\beta_{12}+\alpha_{23}\beta_{23}-\alpha_{31}\beta_{31}) \le$$

$$2\sqrt{-\beta_{12}\gamma_{12}a^2 - \gamma_{12}\alpha_{12}b^2 - \alpha_{12}\beta_{12}c^2} \cdot \sqrt{-\beta_{23}\gamma_{23}a^2 - \gamma_{23}\alpha_{23}b^2 - \alpha_{23}\beta_{23}c^2} \tag{34}$$

Proof. The inequalities (34) are simple direct consequences of the inequalities $-1 \le \cos \widehat{I_1I_2I_3} \le 1$.

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