"Vasile Alecsandri" University of Bacău
Faculty of Sciences
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THE ISOTOMIC TRANSVERSAL THEOREM AND THE NEUBERG'S THEOREM IN THE POINCARÉ MODEL OF HYPERBOLIC GEOMETRY

CĂTĂLIN BARBU

Abstract. In this study, we present a proof of the isotomic transversal theorem and the Neuberg's theorem in hyperbolic geometry

1. Introduction

Hyperbolic geometry appeared in the first half of the 19^{th} century as an attempt to understand Euclid's axiomatic basis of geometry. It is also known as a type of non-euclidean geometry, being in many respects similar to euclidean geometry. Hyperbolic geometry includes similar concepts as distance and angle. Both these geometries have many results in common but many are different. Several useful models of hyperbolic geometry are studied in the literature as, for instance, the Poincaré disc and ball models, the Poincaré half-plane model, and the Beltrami-Klein disc and ball models [1] etc. Following [2] and [3] and earlier discoveries, the Beltrami-Klein model is also known as the Einstein relativistic velocity model. Here, in this study, we give hyperbolic versions of the isotomic transversal theorem and the Neuberg theorem.

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The well-known the isotomic transversal theorem states that if l is a line not through any vertex of a triangle ABC such that l meets sidelines BC, CA, and AB in points A', B', and C', respectively, and let A'' be the reflection of A' about the midpoint of segment BC, and construct B'' and C'' similarly, then A'', B'', and C'' are collinear in a line known as the isotomic transversal of l [4]. The Neuberg theorem states that if three lines from of a triangle ABC, and concurrent at P, meet the opposite sides at P_1, P_2, P_3 respectively; and if we cut off BQ_1, CQ_2, AQ_3 equal respectively P_1C, P_2A, P_3B, then AQ_1, BQ_2, and CQ_3 are concurrent [5]. We use in this study of Poincaré disc model.

We begin with the recall of some basic geometric notions and properties in the Poincaré disc. Let D denote the unit disc in the complex z - plane, i.e.

$$D = \{z \in \mathbb{C} : |z| < 1\}$$

The most general Möbius transformation of D is

$$z \to e^{i\theta} \frac{z_0 + z}{1 + \overline{z_0} z} = e^{i\theta}(z_0 \oplus z),$$

which induces the Möbius addition $\oplus$ in D, allowing the Möbius transformation of the disc to be viewed as a Möbius left gyro-translation

$$z \to z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0} z}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $z, z_0 \in D$, and $\overline{z_0}$ is the complex conjugate of z_0. Let $Aut(D, \oplus)$ be the automorphism group of the grupoid $(D, \oplus)$. If we define

$$gyr : D \times D \to Aut(D, \oplus), gyr[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\overline{b}}{1 + \overline{a}b},$$

then is true gyro-commutative law

$$a \oplus b = gyr[a, b](b \oplus a).$$

A gyro-vector space $(G, \oplus, \otimes)$ is a gyro-commutative gyro-group $(G, \oplus)$ that obeys the following axioms:

(1) $gyr[\mathbf{u}, \mathbf{v}]\mathbf{a} \cdot gyr[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a} \cdot \mathbf{b}$ for all points $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in G$.

(2) G admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $\mathbf{a} \in G$:

(G1) $1 \otimes \mathbf{a} = \mathbf{a}$

(G2) $(r_1 + r_2) \otimes \mathbf{a} = r_1 \otimes \mathbf{a} \oplus r_2 \otimes \mathbf{a}$

(G3) $(r_1 r_2)\otimes \mathbf{a} = r_1 \otimes (r_2 \otimes \mathbf{a})$

(G4) $\frac{|r|\otimes \mathbf{a}}{\|r\otimes \mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$

(G5) $gyr[\mathbf{u},\mathbf{v}](r\otimes \mathbf{a}) = r\otimes gyr[\mathbf{u},\mathbf{v}]\mathbf{a}$

(G6) $gyr[r_1\otimes \mathbf{v}, r_1\otimes \mathbf{v}] = 1$

(3) Real vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of one-dimensional "vectors"

$$\|G\| = \{\pm \|\mathbf{a}\| : \mathbf{a}\in G\} \subset \mathbb{R}$$

with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r\in\mathbb{R}$ and $\mathbf{a},\mathbf{b}\in G$,

(G7) $\|r\otimes \mathbf{a}\| = |r|\otimes \|\mathbf{a}\|$

(G8) $\|\mathbf{a}\oplus \mathbf{b}\| \le \|\mathbf{a}\| \oplus \|\mathbf{b}\|$

Definition 1. *The hyperbolic distance function in D is defined by the equation*

$$d(a,b) = |a\ominus b| = \left|\frac{a-b}{1-\overline{a}b}\right|.$$

Here, $a\ominus b = a\oplus(-b)$, for $a,b\in D$.

For further details we refer to the recent book of A.Ungar [6].

Theorem 2. ***(The Menelaus's theorem for hyperbolic triangle)*** *If l is an gyroline not through any gyrovertex of an gyrotriangle ABC such that l meets BC in D, CA in E, and AB in F, then*

$$\frac{(AF)_\gamma}{(BF)_\gamma}\cdot\frac{(BD)_\gamma}{(CD)_\gamma}\cdot\frac{(CE)_\gamma}{(AE)_\gamma} = 1,$$

where $v_\gamma = \frac{v}{1-v^2}$.

(See [7])

Theorem 3. ***(Converse of Menelaus's theorem for hyperbolic triangle)*** *If D lies on the gyroline BC, E on CA, and F on AB such that*

$$\frac{(AF)_\gamma}{(BF)_\gamma}\cdot\frac{(BD)_\gamma}{(CD)_\gamma}\cdot\frac{(CE)_\gamma}{(AE)_\gamma} = 1,$$

then D, E, and F are collinear.

(See [7])

Theorem 4. ***(The Ceva's theorem for hyperbolic triangle)*** *If M is a point not on any side of a gyrotriangle $A_1A_2A_3$ such that A_3M and A_1A_2 meet in P, A_2M and A_3A_1 in Q, and A_1M and A_2A_3 meet in R, then*

$$\frac{(A_1P)_\gamma}{(A_2P)_\gamma} \cdot \frac{(A_2R)_\gamma}{(A_3R)_\gamma} \cdot \frac{(A_3Q)_\gamma}{(A_1Q)_\gamma} = 1$$

(See [8])

Theorem 5. ***(Converse of Ceva's theorem for hyperbolic triangle)*** *If P lies on the gyroline A_1A_2, R on A_2A_3, and Q on A_3A_1 such that*

$$\frac{(A_1P)_\gamma}{(A_2P)_\gamma} \cdot \frac{(A_2R)_\gamma}{(A_3R)_\gamma} \cdot \frac{(A_3Q)_\gamma}{(A_1Q)_\gamma} = 1,$$

and two of the gyrolines A_1R, A_2Q and A_3P meet, then all three are concurrent.

(See [8])

Theorem 6. ***(The Carnot's theorem for hyperbolic triangle)*** *Let ABC be a hyperbolic triangle in the Poincaré disc, whose are the points A, B and C of the disc and whose sides (directed counterclockwise) are $a = -B \oplus C, b = -C \oplus A$ and $c = -A \oplus B$. Let the points A', B' and C' be located on the sides a, b and c of the hyperbolic triangle ABC respectively. If the perpendiculars to the sides of the hyperbolic triangle at the points A', B' and C' are concurrent, then the following holds:*

$$|-A \oplus C'|^2 \ominus |-B \oplus C'|^2 \oplus |-B \oplus A'|^2 \ominus |-C \oplus A'|^2 \oplus |-C \oplus B'|^2 \ominus |-A \oplus B'|^2 = 0.$$

(See [10])

Theorem 7. ***(Converse of Carnot's theorem for hyperbolic triangle)*** *Let ABC be a hyperbolic triangle in the Poincaré disc, whose are the points A, B and C of the disc and whose sides (directed counterclockwise) are $a = -B \oplus C, b = -C \oplus A$ and $c = -A \oplus B$. Let the points A', B' and C' be located on the sides a, b and c of the hyperbolic triangle ABC respectively. If the following holds*

$$|-A \oplus C'|^2 \ominus |-B \oplus C'|^2 \oplus |-B \oplus A'|^2 \ominus |-C \oplus A'|^2 \oplus |-C \oplus B'|^2 \ominus |-A \oplus B'|^2 = 0,$$

and two of the three perpendiculars to the sides of the hyperbolic triangle at the points A', B' and C' are concurrent, then the three perpendiculars are concurrent.

(See [10])

2. Main results

In this section, we prove the isotomic transversal theorem and Neuberg's theorem for hyperbolic triangle.

Theorem 8. ***(The isotomic transversal theorem).*** *Let l be a gyroline not through any gyrovertex of a gyrotriangle ABC such that l meets gyroside BC, CA, and AB in gyropoints A', B', and C', respectively, and let A'' be the reflection of A' about the midpoint of gyrosegment BC, and construct B'' and C'' similarly, then the gyropoints A'', B'', and C'' are collinear.*

Proof. If we use a theorem 2 for the gytriangle ABC and collinear gyropoints A', B', and C' (See Figure 1), then

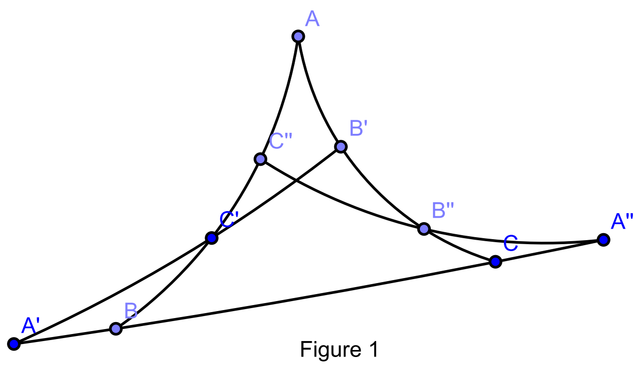

Figure 1

$$\frac{(BA')_\gamma}{(A'C)_\gamma} \cdot \frac{(CB')_\gamma}{(B'A)_\gamma} \cdot \frac{(AC')_\gamma}{(C'B)_\gamma} = 1. \tag{1}$$

Because

$$d(A', B) = d(A'', C), d(A', C) = d(A'', B), d(B', C) = d(B'', A),$$

and

$$d(B', A) = d(B'', C), d(C', A) = d(C'', B), d(C', B) = d(C'', A),$$

we get

$$\frac{(BA')_\gamma}{(A'C)_\gamma} = \frac{(CA'')_\gamma}{(A''B)_\gamma}, \ \frac{(CB')_\gamma}{(B'A)_\gamma} = \frac{(AB'')_\gamma}{(B''C)_\gamma}, \ \frac{(AC')_\gamma}{(C'B)_\gamma} = \frac{(BC'')_\gamma}{(C''A)_\gamma}. \tag{2}$$

From (1) and (2) we have

$$\frac{(CA'')_\gamma}{(A''B)_\gamma} \cdot \frac{(AB'')_\gamma}{(B''C)_\gamma} \cdot \frac{(BC'')_\gamma}{(C''A)_\gamma} = 1,$$

and from the theorem 3, we obtain that the gyropoints A'', B'', and C'' are collinear in a line known as the isotomic transversal of l. ∎

Theorem 9. ***(Neuberg's theorem for hyperbolic triangle).*** *If three gyrolines from of a gyrotriangle ABC, and concurrent at P, meet the opposite gyrosides at P_1, P_2, P_3 respectively; and if we cut off BQ_1, CQ_2, AQ_3 equal respectively P_1C, P_2A, P_3B, and two of the gyrolines AQ_1, BQ_2, and CQ_3 meet, then all three are concurrent.*

Proof. If we use Ceva's theorem in the gyrotriangle ABC (See Theorem 3, Figure 2), then

$$\frac{(BP_1)_\gamma}{(P_1C)_\gamma} \cdot \frac{(CP_2)_\gamma}{(P_2A)_\gamma} \cdot \frac{(AP_3)_\gamma}{(P_3B)_\gamma} = 1. \tag{3}$$

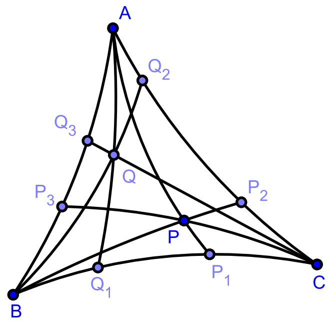

Figure 2

Because of $d(B, P_1) = d(C, Q_1), d(P_1, C) = d(B, Q_1), d(C, P_2) = d(A, Q_2), d(P_2, A) = d(C, Q_2), d(B, P_3) = d(A, Q_3), d(P_3, A) = d(B, Q_3)$, then equations (3) become

$$\frac{(CQ_1)_\gamma}{(Q_1B)_\gamma} \cdot \frac{(AQ_2)_\gamma}{(Q_2C)_\gamma} \cdot \frac{(BQ_3)_\gamma}{(Q_3A)_\gamma} = 1, \tag{4}$$

and by Theorem 4 we obtain that the gyrolines AQ_1, BQ_2, and CQ_3 are concurrent in a gyropoint Q, called the isotomic conjugate of P. ∎

Corollary 10. *The centroid G of a gyrotriangle is its own isotomic conjugate of G.*

Corollary 11. *If exists the Nagel point of the gyrotriangle ABC, then the Gergonne gyropoint is your isotomic conjugate.*

Theorem 12. *Let ABC be a gyrotriangle in the Poincaré disc, whose are the gyropoints A, B and C of the disc and whose sides (directed counterclockwise) are $a = -B \oplus C, b = -C \oplus A$ and $c = -A \oplus B$. Let the gyropoints A', B' and C' be located on the gyrosides a, b and c of the gyrotriangle ABC respectively, and let A'' be the reflection of A' about the midpoint of gyrosegment BC, and construct B'' and C'' similarly. If the perpendiculars to the gyrosides of the gyrotriangle at the points A', B' and C' are concurrent, and two of the three perpendiculars to the sides of the hyperbolic triangle at the points A'', B'' and C'' are concurrent, then the three perpendiculars are concurrent.*

Proof. If we use Theorem 6 in the gyrotriangle ABC (See Figure 3), then
(5)
$$|-A \oplus C'|^2 \ominus |-B \oplus C'|^2 \oplus |-B \oplus A'|^2 \ominus |-C \oplus A'|^2 \oplus |-C \oplus B'|^2 \ominus |-A \oplus B'|^2 = 0.$$

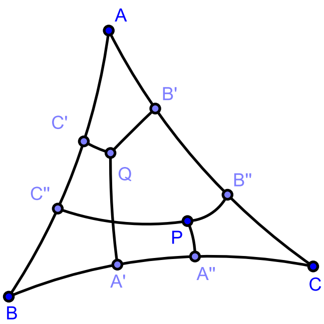

Figure 3

Because $|-A \oplus C'| = |-B \oplus C''|, |-B \oplus C'| = |-A \oplus C''|, |-B \oplus A'| = |-C \oplus A''|, |-C \oplus A'| = |-B \oplus A''|, |-C \oplus B'| = |-A \oplus B''|, |-A \oplus B'| = |-C \oplus B''|,$ then (5) become

$$|-B \oplus C''|^2 \ominus |-A \oplus C''|^2 \oplus |-C \oplus A''|^2 \ominus |-B \oplus A''|^2 \oplus |-A \oplus B''|^2 \ominus |-C \oplus B''|^2 = 0,$$

and by Theorem 7 we obtain the conlusion. ∎

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"Vasile Alecsandri" College
Bacău, Romania
e-mail: kafka_mate@yahoo.com