Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.1, Issue2, pp.71-74, Galley Proofs

THE ORTHOPOLE THEOREM IN THE POINCARÉ UPPER HALF-PLANE OF HYPERBOLIC GEOMETRY

CĂTĂLIN BARBU AND NILGÜN SÖNMEZ

2010 Mathematical Subject Classification:30F45, 51M10.
Keywords and phrases:hyperbolic geometry, hyperbolic triangle, Soons's theorem, Poincaré upper half-plane model.

ABSTRACT. In this study we prove the orthopole theorem for a hyperbolic triangle.

1. INTRODUCTION

Hyperbolic geometry appeared in the first half of the 19^{th} century as an attempt to understand Euclid's axiomatic basis of geometry. It is also known as a type of non-euclidean geometry, being in many respects similar to euclidean geometry. Hyperbolic geometry includes similar concepts as distance and angle. Both these geometries have many results in common but many are different. Several useful models of hyperbolic geometry are studied in the literature as, for instance, the Poincaré disc and ball models, the Poincaré half-plane model, and the Beltrami-Klein disc and ball models [7] etc. Here, in this study, we give hyperbolic version of the orthopole theorem in the Poincaré upper half-plane of hyperbolic geometry. The well-known the orthopole theorem states that if A', B', C' be the projections of the vertices A, B, C of the triangle ABC on a straight line d, the perpendiculars from A' on BC, from B' on CA, and from C' on AB are concurrent at a point called the orthopole of d for the triangle ABC [5]. This result has a simple statement but it is of great interes. We just mention here few different proofs given by W. Gallaty [3], R. Goormaghtigh [4], J. Neuberg [6].
We mention that C. Barbu and L. Pişcoran [1] gave the hyperbolic form of Soons's theorem in the Poincaré disc model of hyperbolic geometry. In order to introduce the Carnot's theorem into the Poincaré upper half-plane we refer briefly some facts about the Poincaré upper half-plane.

2. Preliminaries

The nature of the x-axis is such as to make impossible any communication between the lower and the upper half-planes. We restrict our attention to the upper half-plane and refer to it as the hyperbolic plane. It is also known as the Poincaré upper half-plane. The geodesic segments of the Poincaré upper half-plane (hyperbolic plane) are either segments of Euclidean straight lines that are perpendicular to the x-axis or arc of Euclidean

semicircles that are centered on the x-axis. The hyperbolic length of the Euclidean line segment joining the points $P = (a; y_1)$ and $Q = (a; y_2)$, $0 < y_1 \leq y_2$, is $\ln \frac{y_2}{y_1}$.
The hyperbolic length between the points P and Q on a Euclidean semicircle with center $C = (c; 0)$ and radius r such that the radii CP and CQ make angles α and β ($\alpha < \beta$) respectively, with the positive x-axis [8],

$$\ln \frac{\csc \beta - \cot \beta}{\csc \alpha - \cot \alpha}.$$

Theorem 1.1. Let ABC be a hyperbolic triangle with a right angle at C. If a, b, c, are the hyperbolic lengths of the sides opposite A, B, C, respectively, then

$$\cosh c = \cosh a \cdot \cosh b.$$

For the proof of the theorem see [8].

Theorem 1.2. Let ABC be a hyperbolic triangle. Let the points A', B', and C' be located on the sides BC, CA and AB of the hyperbolic triangle ABC respectively. If the perpendiculars to the sides of the hyperbolic triangle at the points B' and C' are concurrent in the point M and the following relation holds

$$\frac{\cosh A'B}{\cosh A'C} \cdot \frac{\cosh B'C}{\cosh B'A} \cdot \frac{\cosh C'A}{\cosh C'B} = 1,$$

then the point M is on the perpendicular to BC at the point A'.

For the proof of the theorem see [2].

3. The hyperbolic Soons theorem in the Poincaré upper half-plane model of hyperbolic geometry

In this section, we prove the orthopole theorem for a hyperbolic triangle.

Theorem 1.3. Let A', B', C' be the projections of the vertices A, B, C of the gyrotriangle ABC on a straight gyroline d. If two of the three perpendiculars from A' on BC, from B' on CA, and from C' on AB are concurrent, then the three perpendiculars are concurrent.

Proof. Let's note A'', B'', C'' the projections of the points A', B', C' on BC, CA, AB, respectively (See Figure 1).

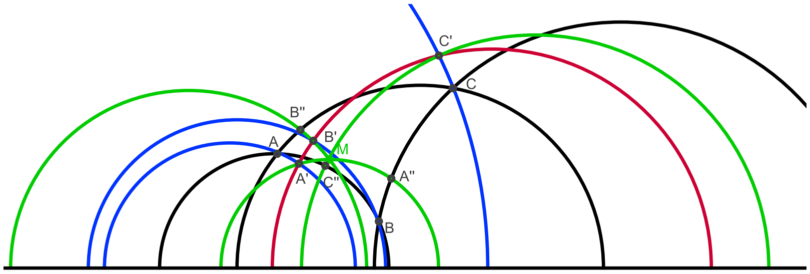

Figure 1

If we use the Theorem 2 in the gyrotriangles $AA'B'$ and $AA'C'$, we get

$$\cosh AB' = \cosh B'A' \cdot \cosh A'A \tag{1}$$

and

$$\cosh AC' = \cosh C'A' \cdot \cosh A'A \tag{2}$$

By the relations (1) and (2), we have

$$\frac{\cosh AB'}{\cosh AC'} = \frac{\cosh B'A'}{\cosh C'A'} \tag{3}$$

Similary we prove that

$$\frac{\cosh BC'}{\cosh BA'} = \frac{\cosh C'B'}{\cosh A'B'}, \tag{4}$$

respectively

$$\frac{\cosh CA'}{\cosh CB'} = \frac{\cosh A'C'}{\cosh B'C'}. \tag{5}$$

By the relations (3), (4) and (5) we get

$$\frac{\cosh AB'}{\cosh AC'} \cdot \frac{\cosh BC'}{\cosh BA'} \cdot \frac{\cosh CA'}{\cosh CB'} = \frac{\cosh B'A'}{\cosh C'A'} \cdot \frac{\cosh C'B'}{\cosh A'B'} \cdot \frac{\cosh A'C'}{\cosh B'C'} = 1. \tag{6}$$

If we use the Theorem 1 in the gyrotriangles $AB'B'', AC'C'', BC'C'', BA'A'', CA'A''$ and $CB'B''$, we get

$$\cosh AB' = \cosh B'B'' \cdot \cosh B''A, \tag{7}$$

$$\cosh AC' = \cosh C'C'' \cdot \cosh C''A, \tag{8}$$

$$\cosh BC' = \cosh C'C'' \cdot \cosh C''B, \tag{9}$$

$$\cosh BA' = \cosh A'A'' \cdot \cosh A''B, \tag{10}$$

$$\cosh CA' = \cosh A'A'' \cdot \cosh A''C, \tag{11}$$

$$\cosh CB' = \cosh B'B'' \cdot \cosh B''C. \tag{12}$$

By the relations (7) and (8), result

$$\frac{\cosh AB'}{\cosh AC'} = \frac{\cosh B'B''}{\cosh C'C''} \cdot \frac{\cosh B''A}{\cosh C''A}. \tag{13}$$

By the relations (9) and (10), we obtain

$$\frac{\cosh BC'}{\cosh BA'} = \frac{\cosh C'C''}{\cosh A'A''} \cdot \frac{\cosh C''B}{\cosh A''B}, \tag{14}$$

and by the relations (11) and (12), we get

$$\frac{\cosh CA'}{\cosh CB'} = \frac{\cosh A'A''}{\cosh B'B''} \cdot \frac{\cosh A''C}{\cosh B''C}. \tag{15}$$

Multiplying the relations (13), (14) and (15) member by member, and we use (6), we obtain

$$\begin{gathered}
1 = \frac{\cosh AB'}{\cosh AC'} \cdot \frac{\cosh BC'}{\cosh BA'} \cdot \frac{\cosh CA'}{\cosh CB'} = \\
\left(\frac{\cosh B'B''}{\cosh C'C''} \cdot \frac{\cosh C'C''}{\cosh A'A''} \cdot \frac{\cosh A'A''}{\cosh B'B''}\right) \cdot \left(\frac{\cosh B''A}{\cosh C''A} \cdot \frac{\cosh C''B}{\cosh A''B} \cdot \frac{\cosh A''C}{\cosh B''C}\right) = \\
\frac{\cosh B''A}{\cosh C''A} \cdot \frac{\cosh C''B}{\cosh A''B} \cdot \frac{\cosh A''C}{\cosh B''C}
\end{gathered}$$

and by Theorem 2 we obtain that the gyrolines $A'A''$, $B'B''$, and $C'C''$ are concurrent. ∎

REFERENCES

[1] Barbu, C., Pişcoran, L., *The orthopole theorem in the Poincaré disc model of hyperbolic geometry*, Acta Universitatis Sapientia, accepted.

[2] Barbu, C., Sönmez, N., *On the Carnot theorem in the Poincaré upper half-plane model of hyperbolic geometry*, (subbmited).

[3] Gallaty, W., *The Modern Geometry of the Triangle,* Hodgson Pub., London, 1922, p.46.

[4] Goormaghtigh, R., *A generalization of the orthopole theorem*, Vol. 36, 1929, pp.422-424.

[5] Johnson, R., *A. Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle. Boston*, MA: Houghton Mifflin, p. 247, 1929.

[6] Neuberg, J., *Nouvelle Correspondance Mathématique*, 1875, p.189.

[7] McCleary, J., *Geometry from a differentiable viewpoint,* Cambridge University Press, Cambridge, 1994.

[8] Stahl, S., *The Poincare half plane a gateway to modern geometry*, Jones and Barlett Publishers, Boston, (1993) p. 298.

DEPARTMENT OF MATHEMATICS
VASILE ALECSANDRI NATIONAL COLLEGE
STR. VASILE ALECSANDRI NR. 37, 600011
BACĂU, ROMANIA
E-mail address: `kafka_mate@yahoo.com`

AFYON KOCATEPE UNIVERSITY
FACULTY OF SCIENCE AND LITERATURES
DEPARTMENT OF MATHEMATICS
ANS CAMPUS, 03200 - AFYONKARAHISAR, TURKEY
E-mail address: `nceylan@aku.edu.tr`