Smarandache's Cevian Triangle Theorem in The Einstein Relativistic Velocity Model of Hyperbolic Geometry

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In this note, we present a proof of Smarandache's cevian triangle hyperbolic theorem in the Einstein relativistic velocity model of hyperbolic geometry.

1 Introduction

Hyperbolic geometry appeared in the first half of the 19^{th} century as an attempt to understand Euclid's axiomatic basis for geometry. It is also known as a type of non-Euclidean geometry, being in many respects similar to Euclidean geometry. Hyperbolic geometry includes such concepts as: distance, angle and both of them have many theorems in common.There are known many main models for hyperbolic geometry, such as: Poincaré disc model, Poincaré half-plane, Klein model, Einstein relativistic velocity model, etc. The hyperbolic geometry is a non-Euclidian geometry. Here, in this study, we present a proof of Smarandache's cevian triangle hyperbolic theorem in the Einstein relativistic velocity model of hyperbolic geometry. Smarandache's cevian triangle theorem states that if $A_1B_1C_1$ is the cevian triangle of point P with respect to the triangle ABC, then $\frac{PA}{PA_1} \cdot \frac{PB}{PB_1} \cdot \frac{PC}{PC_1} = \frac{AB \cdot BC \cdot CA}{A_1B \cdot B_1C \cdot C_1A}$ [1].

Let D denote the complex unit disc in complex z - plane, i.e.

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

The most general Möbius transformation of D is

$$z \to e^{i\theta} \frac{z_0 + z}{1 + \bar{z}_0 z} = e^{i\theta}(z_0 \oplus z),$$

which induces the Möbius addition $\oplus$ in D, allowing the Möbius transformation of the disc to be viewed as a Möbius left gyrotranslation

$$z \to z_0 \oplus z = \frac{z_0 + z}{1 + \bar{z}_0 z}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $z, z_0 \in D$, and $\bar{z}_0$ is the complex conjugate of z_0. Let $Aut(D, \oplus)$ be the automorphism group of the grupoid $(D, \oplus)$. If we define

$$gyr : D \times D \to Aut(D, \oplus), \quad gyr[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\bar{b}}{1 + \bar{a}b},$$

then is true gyrocommutative law

$$a \oplus b = gyr[a, b](b \oplus a).$$

A gyrovector space $(G, \oplus, \otimes)$ is a gyrocommutative gyrogroup $(G, \oplus)$ that obeys the following axioms:

(1) $gyr[\mathbf{u}, \mathbf{v}]\mathbf{a} \cdot gyr[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a} \cdot \mathbf{b}$ for all points $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in G$;

(2) G admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $\mathbf{a} \in G$:

G1 $1 \otimes \mathbf{a} = \mathbf{a}$,

G2 $(r_1 + r_2) \otimes \mathbf{a} = r_1 \otimes \mathbf{a} \oplus r_2 \otimes \mathbf{a}$,

G3 $(r_1 r_2) \otimes \mathbf{a} = r_1 \otimes (r_2 \otimes \mathbf{a})$,

G4 $\frac{|r| \otimes \mathbf{a}}{\|r \otimes \mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$,

G5 $gyr[\mathbf{u}, \mathbf{v}](r \otimes \mathbf{a}) = r \otimes gyr[\mathbf{u}, \mathbf{v}]\mathbf{a}$,

G6 $gyr[r_1 \otimes \mathbf{v}, r_1 \otimes \mathbf{v}] = 1$;

(3) Real vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of onedimensional "vectors"

$$\|G\| = \{\pm \|\mathbf{a}\| : \mathbf{a} \in G\} \subset \mathbb{R}$$

with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in G$:

G7 $\|r \otimes \mathbf{a}\| = |r| \otimes \|\mathbf{a}\|$,

G8 $\|\mathbf{a} \oplus \mathbf{b}\| \leqslant \|\mathbf{a}\| \oplus \|\mathbf{b}\|$.

Theorem 1 The Hyperbolic Theorem of Ceva in Einstein Gyrovector Space *Let $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a}_3$ be three non-gyrocollinear points in an Einstein gyrovector space $(V_s, \oplus, \otimes)$. Furthermore, let $\mathbf{a}_{123}$ be a point in their gyroplane, which is off the gyrolines $\mathbf{a}_1\mathbf{a}_2$, $\mathbf{a}_2\mathbf{a}_3$, and $\mathbf{a}_3\mathbf{a}_1$. If $\mathbf{a}_1\mathbf{a}_{123}$ meets $\mathbf{a}_2\mathbf{a}_3$ at $\mathbf{a}_{23}$, etc., then*

$$\frac{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}} \|\ominus\mathbf{a}_1 \oplus \mathbf{a}_{12}\|}{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}} \|\ominus\mathbf{a}_2 \oplus \mathbf{a}_{12}\|} \cdot \frac{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{23}} \|\ominus\mathbf{a}_2 \oplus \mathbf{a}_{23}\|}{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{23}} \|\ominus\mathbf{a}_3 \oplus \mathbf{a}_{23}\|} \times \\ \times \frac{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{13}} \|\ominus\mathbf{a}_3 \oplus \mathbf{a}_{13}\|}{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{13}} \|\ominus\mathbf{a}_1 \oplus \mathbf{a}_{13}\|} = 1,$$

(here $\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{v}\|^2}{s^2}}}$ is the gamma factor). (See [2, pp. 461].)

Theorem 2 The Hyperbolic Theorem of Menelaus in Einstein Gyrovector Space *Let $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a}_3$ be three non-gyrocollinear points in an Einstein gyrovector space $(V_s, \oplus, \otimes)$. If a gyroline meets the sides of gyrotriangle $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$ at points $\mathbf{a}_{12}, \mathbf{a}_{13}, \mathbf{a}_{23}$, then*

$$\frac{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}} \|\ominus\mathbf{a}_1 \oplus \mathbf{a}_{12}\|}{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}} \|\ominus\mathbf{a}_2 \oplus \mathbf{a}_{12}\|} \cdot \frac{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{23}} \|\ominus\mathbf{a}_2 \oplus \mathbf{a}_{23}\|}{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{23}} \|\ominus\mathbf{a}_3 \oplus \mathbf{a}_{23}\|} \times \\ \times \frac{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{13}} \|\ominus\mathbf{a}_3 \oplus \mathbf{a}_{13}\|}{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{13}} \|\ominus\mathbf{a}_1 \oplus \mathbf{a}_{13}\|} = 1.$$

(See [2, pp. 463].) For further details we refer to A. Ungar's recent book [2].

2 Main result

In this section, we present a proof of Smarandache's cevian triangle hyperbolic theorem in the Einstein relativistic velocity model of hyperbolic geometry.

Theorem 3 *If $A_1B_1C_1$ is the cevian gyrotriangle of gyropoint P with respect to the gyrotriangle ABC, then*

$$\frac{\gamma_{|PA|}|PA|}{\gamma_{|PA_1|}|PA_1|}\cdot\frac{\gamma_{|PB|}|PB|}{\gamma_{|PB_1|}|PB_1|}\cdot\frac{\gamma_{|PC|}|PC|}{\gamma_{|PC_1|}|PC_1|}=\frac{\gamma_{|AB|}|AB|\cdot\gamma_{|BC|}|BC|\cdot\gamma_{|CA|}|CA|}{\gamma_{|AB_1|}|AB_1|\cdot\gamma_{|BC_1|}|BC_1|\cdot\gamma_{|CA_1|}|CA_1|}.$$

Proof *If we use a theorem 2 in the gyrotriangle ABC (see Figure), we have*

$$\gamma_{|AC_1|}|AC_1|\cdot\gamma_{|BA_1|}|BA_1|\cdot\gamma_{|CB_1|}|CB_1|=\gamma_{|AB_1|}|AB_1|\cdot\gamma_{|BC_1|}|BC_1|\cdot\gamma_{|CA_1|}|CA_1|. \tag{1}$$

If we use a theorem 1 in the gyrotriangle AA_1B, cut by the gyroline CC_1, we get

$$\gamma_{|AC_1|}|AC_1|\cdot\gamma_{|BC|}|BC|\cdot\gamma_{|A_1P|}|A_1P|=\gamma_{|AP|}|AP|\cdot\gamma_{|A_1C|}|A_1C|\cdot\gamma_{|BC_1|}|BC_1|. \tag{2}$$

If we use a theorem 1 in the gyrotriangle BB_1C, cut by the gyroline AA_1, we get

$$\gamma_{|BA_1|}|BA_1|\cdot\gamma_{|CA|}|CA|\cdot\gamma_{|B_1P|}|B_1P|=\gamma_{|BP|}|BP|\cdot\gamma_{|B_1A|}|B_1A|\cdot\gamma_{|CA_1|}|CA_1|. \tag{3}$$

If we use a theorem 1 in the gyrotriangle CC_1A, cut by the gyroline BB_1, we get

$$\gamma_{|CB_1|}|CB_1|\cdot\gamma_{|AB|}|AB|\cdot\gamma_{|C_1P|}|C_1P|=\gamma_{|CP|}|CP|\cdot\gamma_{|C_1B|}|C_1B|\cdot\gamma_{|AB_1|}|AB_1|. \tag{4}$$

We divide each relation (2), (3), and (4) by relation (1), and we obtain

$$\frac{\gamma_{|PA|}|PA|}{\gamma_{|PA_1|}|PA_1|}=\frac{\gamma_{|BC|}|BC|}{\gamma_{|BA_1|}|BA_1|}\cdot\frac{\gamma_{|B_1A|}|B_1A|}{\gamma_{|B_1C|}|B_1C|}, \tag{5}$$

$$\frac{\gamma_{|PB|}|PB|}{\gamma_{|PB_1|}|PB_1|}=\frac{\gamma_{|CA|}|CA|}{\gamma_{|CB_1|}|CB_1|}\cdot\frac{\gamma_{|C_1B|}|C_1B|}{\gamma_{|C_1A|}|C_1A|}, \tag{6}$$

$$\frac{\gamma_{|PC|}|PC|}{\gamma_{|PC_1|}|PC_1|}=\frac{\gamma_{|AB|}|AB|}{\gamma_{|AC_1|}|AC_1|}\cdot\frac{\gamma_{|A_1C|}|A_1C|}{\gamma_{|A_1B|}|A_1B|}. \tag{7}$$

Multiplying (5) by (6) and by (7), we have

$$\frac{\gamma_{|PA|}|PA|}{\gamma_{|PA_1|}|PA_1|}\cdot\frac{\gamma_{|PB|}|PB|}{\gamma_{|PB_1|}|PB_1|}\cdot\frac{\gamma_{|PC|}|PC|}{\gamma_{|PC_1|}|PC_1|}=$$
$$=\frac{\gamma_{|AB|}|AB|\cdot\gamma_{|BC|}|BC|\cdot\gamma_{|CA|}|CA|}{\gamma_{|A_1B|}|A_1B|\cdot\gamma_{|B_1C|}|B_1C|\cdot\gamma_{|C_1A|}|C_1A|}\cdot\frac{\gamma_{|B_1A|}|B_1A|\cdot\gamma_{|C_1B|}|C_1B|\cdot\gamma_{|A_1C|}|A_1C|}{\gamma_{|A_1B|}|A_1B|\cdot\gamma_{|B_1C|}|B_1C|\cdot\gamma_{|C_1A|}|C_1A|}. \tag{8}$$

From the relation (1) we have

$$\frac{\gamma_{|B_1A|}|B_1A|\cdot\gamma_{|C_1B|}|C_1B|\cdot\gamma_{|A_1C|}|A_1C|}{\gamma_{|A_1B|}|A_1B|\cdot\gamma_{|B_1C|}|B_1C|\cdot\gamma_{|C_1A|}|C_1A|}=1, \tag{9}$$

so

$$\frac{\gamma_{|PA|}|PA|}{\gamma_{|PA_1|}|PA_1|}\cdot\frac{\gamma_{|PB|}|PB|}{\gamma_{|PB_1|}|PB_1|}\cdot\frac{\gamma_{|PC|}|PC|}{\gamma_{|PC_1|}|PC_1|}=\frac{\gamma_{|AB|}|AB|\cdot\gamma_{|BC|}|BC|\cdot\gamma_{|CA|}|CA|}{\gamma_{|AB_1|}|AB_1|\cdot\gamma_{|BC_1|}|BC_1|\cdot\gamma_{|CA_1|}|CA_1|}.$$

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References

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