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THE GEOMETRY OF BLUNDON'S CONFIGURATION

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Abstract. Denote by $\mathscr{T}(R,r)$ the family of triangles inscribed in the circle of center O with the radius R and circumscribed to the circle of center I with the radius r. This defines the Blundon's configuration. The family $\mathscr{T}(R,r)$ contains only two isosceles triangles $A_{\min}B_{\min}C_{\min}$ and $A_{\max}B_{\max}C_{\max}$, which are extremal for Blundon's inequalites (1). Some properties of Blundon's configuration are given Section 2. Applications are presented in the last section where a strong version of Blundon's inequalites is obtained (Theorem 7).

1. Introduction

Given a triangle ABC, denote by O the circumcenter, I the incenter, N the Nagel point, s the semiperimeter, R the circumradius, and r the inradius of ABC. W. J. Blundon [7] has proved in 1965 that the following inequalities hold

$$2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr} \leqslant s^2 \leqslant 2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}. \tag{1}$$

The inequalities (1) are fundamental in triangle geometry because they represent necessary and sufficient conditions (see [7]) for the existence of a triangle with given elements R,r and s. The algebraic character of inequalities (1) is discussed in the papers [10] and [11] and an elementary proof to the weak form of (1) is given in [8]. Other results connected to (1) are contained in [13]. We mention that D. Andrica, C. Barbu [2] (see also [1, Section 4.6.5, pp.125-127]) give a direct geometric proof to Blundon's inequalities by using the Law of Cosines in triangle ION. They have obtained the formula

$$\cos\widehat{ION} = \frac{2R^2+10Rr-r^2-s^2}{2(R-2r)\sqrt{R^2-2Rr}}. \tag{2}$$

Because $-1 \leqslant \cos\widehat{ION} \leqslant 1$, obviously it follows that (2) implies (1), showing the geometric character of (1). In the paper [3] other Blundon's type inequalities are obtained using the same idea and different points instead of points I,O,N. If ϕ denotes $\min\{|A-B|,|B-C|,|C-A|\}$, then in the paper [15] is proved the following improvement to (1), $-\cos\phi \leqslant \cos\widehat{ION} \leqslant \cos\phi$. A geometric proof to this inequalities is given in the paper [4].

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In Section 2 of the present note we study some geometric properties of the Blundon's configuration. In the last section we present a strong version of Blundon's inequalities.

2. The Blundon's configuration

It is well-known that distance between points O and N is given by

$$ON = R - 2r. \tag{3}$$

The relation (3) reflects geometrically the difference between the quantities involved in the Euler's inequality $R \geqslant 2r$. In the book of T.Andreescu and D.Andrica [1, Theorem 1, pp.122-123] is given a proof to relation (3) using complex numbers. In the paper [5] similar relations involving the circumradius and the exradii of the triangle are proved and discussed.

Denote by $\mathscr{T}(R,r)$ the family of all triangles having the circumradius R and the inradius r, inscribed in the circle of center O and circumscribed to the circle of center I, where the points O and I are fixed. Let us observe that the inequalities (1) give in terms of R and r the exact interval containing the semiperimeter s for triangles in family $\mathscr{T}(R,r)$.

More exactly, we have

$$s^2_{\min} = 2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R^2 - 2Rr}$$

and

$$s^2_{\max} = 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R^2 - 2Rr}.$$

The triangles in the family $\mathscr{T}(R,r)$ are situated "between" two extremal triangles $A_{\min}B_{\min}C_{\min}$ and $A_{\max}B_{\max}C_{\max}$ determined by $s_{\min}$ and $s_{\max}$. These triangles are isosceles with respect to the vertices $A_{\min}$ and $A_{\max}$. Indeed, according to formula (2), the triangle in the family $\mathscr{T}(R,r)$ with minimal semiperimeter corresponds to the equality case $\cos\widehat{ION} = 1$, i.e. the points I,O,N are collinear and I and N belong to the same ray with the origin O. Let G and H be the centroid and the orthocenter of triangle. Taking in to account the well-known property that points O,G,H belong to Euler's line of triangle, this implies that O,I,G must be collinear, hence in this case triangle ABC is isosceles. In similar way, the triangle in the family $\mathscr{T}(R,r)$ with maximal semiperimeter corresponds to the equality case $\cos\widehat{ION} = -1$, i.e. the points I,O,N are collinear and O is situated between I and N. Using again the Euler's line of the triangle, it follows that triangle ABC is isosceles.

We call the *Blundon's configuration*, the geometric situation in Figure 1.

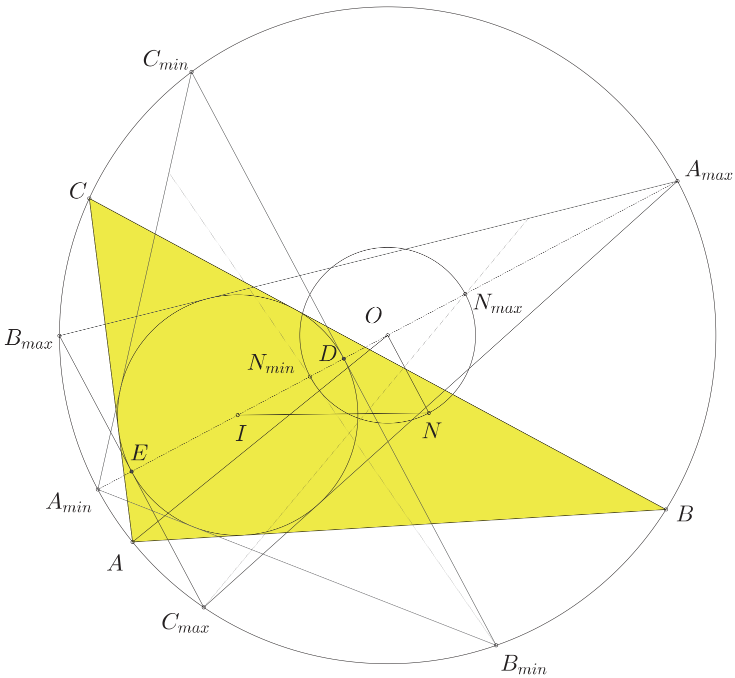

Figure 1. The Blundon's configuration and the Nagel's point N

THEOREM 1. *The family* $\mathscr{T}(R,r)$ *contains only two isosceles triangles, i.e. the extremal triangles* $A_{\min}B_{\min}C_{\min}$ *and* $A_{\max}B_{\max}C_{\max}$.

Proof. The triangle ABC in $\mathscr{T}(R,r)$ is isosceles with $AB = AC$ if and only if OI is perpendicular to BC. Because $B_{\min}C_{\min}$ and $B_{\max}C_{\max}$ are perpendicular to OI, the conclusion follows. □

In what follows we will determine some elements of the isosceles triangles $A_{\min}B_{\min}C_{\min}$ and $A_{\max}B_{\max}C_{\max}$.

We have $A_{\min}D = R - OD = R - (OI - r)$, where the point D is defined in Figure 1. It follows

$$A_{\min}D = h_{\min} = R + r - OI = R + r - \sqrt{R^2 - 2Rr}. \tag{4}$$

Similarly, we have

$$A_{\max}E = h_{\max} = R + r + OI = R + r + \sqrt{R^2 - 2Rr}. \tag{5}$$

REMARK 1. Because $OD \geqslant 0$, it follows $OI \geqslant r$ and we get

$$R \geqslant r(1 + \sqrt{2}), \tag{6}$$

i.e.

$$r \leqslant (\sqrt{2} - 1)R.$$

This is a short geometric proof to the A.Emmerich inequality [9], true for every non-acute triangle.

Consider $a_m = B_{\min}C_{\min}$, $b_m = A_{\min}B_{\min} = A_{\min}C_{\min}$, $K_m = \frac{a_m \cdot h_{\min}}{2}$ the area of triangle $A_{\min}B_{\min}C_{\min}$. We have

$$R = \frac{a_m b_m^2}{4K_m} = \frac{b_m^2}{2h_{\min}},$$

therefore

$$2Rh_{\min} = b_m^2 = h_{\min}^2 + \frac{a_m^2}{4},$$

hence

$$a_m^2 = 4h_{\min}(2R - h_{\min}). \tag{7}$$

From equations (4) and (7) it follows

$$a_m^2 = 4r\left(2R - r + 2\sqrt{R^2 - 2Rr}\right). \tag{8}$$

Denote $a_M = B_{\max}C_{\max}$, $b_M = A_{\max}B_{\max} = A_{\max}C_{\max}$, and let $K_M = \frac{a_M \cdot h_{\max}}{2}$ be the area of triangle $A_{\max}B_{\max}C_{\max}$. We have

$$R = \frac{a_M b_M^2}{4K_M} = \frac{b_M^2}{2h_{\max}},$$

hence

$$2Rh_{\max} = b_M^2 = h_{\max}^2 + \frac{a_M^2}{4}.$$

From here we obtain

$$a_M^2 = 4h_{\max}(2R - h_{\max}). \tag{9}$$

Using the equations (5) and (9) it follows

$$a_M^2 = 4r\left(2R - r - 2\sqrt{R^2 - 2Rr}\right). \tag{10}$$

Combining the equations (8) and (10) we obtain

$$a_m^2 + a_M^2 = 8r(2R - r) \text{ and } a_m a_M = 4r\sqrt{r^2 + 4Rr}.$$

From equations (8) and (10) we get the inequality $a_M < a_m$. Also, we have

$$\cos A_{\min} = 2\cos^2\frac{A_{\min}}{2} - 1 = 2 \cdot \frac{h_{\min}^2}{b_m^2} - 1 = \frac{h_{\min}}{R} - 1, \tag{11}$$

and similarly

$$\cos A_{\max} = 2\cos^2\frac{A_{\max}}{2} - 1 = 2 \cdot \frac{h_{\max}^2}{b_m^2} - 1 = \frac{h_{\max}}{R} - 1. \tag{12}$$

THEOREM 2. *The following relations hold:*

$$\sin\frac{A_{\max}}{2}=\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{2r}{R}} \tag{13}$$

and

$$\sin\frac{A_{\min}}{2}=\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{2r}{R}}. \tag{14}$$

Proof. Using formulas (12) and (5), we have successively

$$\sin^2\frac{A_{\max}}{2}=\frac{1-\cos A_{\max}}{2}=\frac{2-\frac{h_{\max}}{R}}{2}=1-\frac{h_{\max}}{2R}=1-\frac{R+r+\sqrt{R^2-2Rr}}{2R}$$
$$=\frac{R-r-\sqrt{R^2-2Rr}}{2R}=\frac{2R^2-2Rr-2R\sqrt{R^2-2Rr}}{4R^2}=\left(\frac{R-\sqrt{R^2-2Rr}}{2R}\right)^2,$$

and the formula (13) follows.

In similar way, using formulas (11) and (4), we obtain

$$\sin^2\frac{A_{\min}}{2}=\frac{1-\cos A_{\min}}{2}=\frac{2-\frac{h_{\min}}{R}}{2}=1-\frac{h_{\min}}{2R}=1-\frac{R+r-\sqrt{R^2-2Rr}}{2R}$$
$$=\frac{R-r+\sqrt{R^2-2Rr}}{2R}=\frac{2R^2-2Rr+2R\sqrt{R^2-2Rr}}{4R^2}=\left(\frac{R+\sqrt{R^2-2Rr}}{2R}\right)^2,$$

and we get the formula (14). □

The results in Theorem 1 and Theorem 2 clarify with different proofs the results contained in Theorems 1-2 in the paper [14].

3. Consequences for Blundon's inequalities

In this section we give some applications in the spirit of papers [6] and [12]. We begin with the following auxiliary result.

LEMMA 3. *Let P be a point situated in the interior of the circle $\mathscr{C}(O;R)$. If $P\neq O$, then the function $A\mapsto PA$ is strictly increasing on the semicircle $\widehat{M_0M_1}$, where the points M_0,M_1 are the intersection of OP with the circle $\mathscr{C}$ such that $P\in(OM_0)$.*

Proof. Without loss of generality, we can assume that O is the origin of the coordinates system xOy and P is situated on the positive half axis. In this case we have $P(x_0,0), x_0>0, A(R\cos t, R\sin t), t\in[0,\pi]$, and

$$PA^2=(R\cos t-x_0)^2+(R\sin t)^2=R^2+x_0^2-2Rx_0\cos t.$$

Because the cosine function is strictly decreasing on the interval $[0,\pi]$ and $x_0>0$ we obtain that the function $A\mapsto PA^2$ is strictly increasing, and the conclusion follows. □

THEOREM 4. *In the Blundon's configuration, the function* $A \mapsto \angle BAC$ *is strictly increasing on the semicircle* $\widehat{A_{\max}A_{\min}}$.

Proof. We use the well-know relation $\sin\frac{A}{2} = \frac{r}{IA}$. From Lemma 3 with $P = I$, the function $A \mapsto IA$ is strictly decreasing on the semicircle $\widehat{A_{\max}A_{\min}}$. Therefore, for two points $A_1, A_2 \in \widehat{A_{\max}A_{\min}}$ in this order, we have $IA_1 > IA_2$. Therefore $\sin\frac{A_1}{2} = \frac{r}{IA_1} < \frac{r}{IA_2} = \sin\frac{A_2}{2}$, implying $\angle B_1A_1C_1 < \angle B_2A_2C_2$. $\square$

From the Law of Sines, for a triangle in the family $\mathscr{T}(R,r)$, we have $a = 2R\sin A$. Using the relation $r = (s-a)\tan\frac{A}{2}$ we obtain

$$s = \frac{r + a\tan\frac{A}{2}}{\tan\frac{A}{2}} = \frac{r + 2R\sin A\tan\frac{A}{2}}{\tan\frac{A}{2}}, \tag{15}$$

i.e. the semiperimeter s depends only on the angle A.

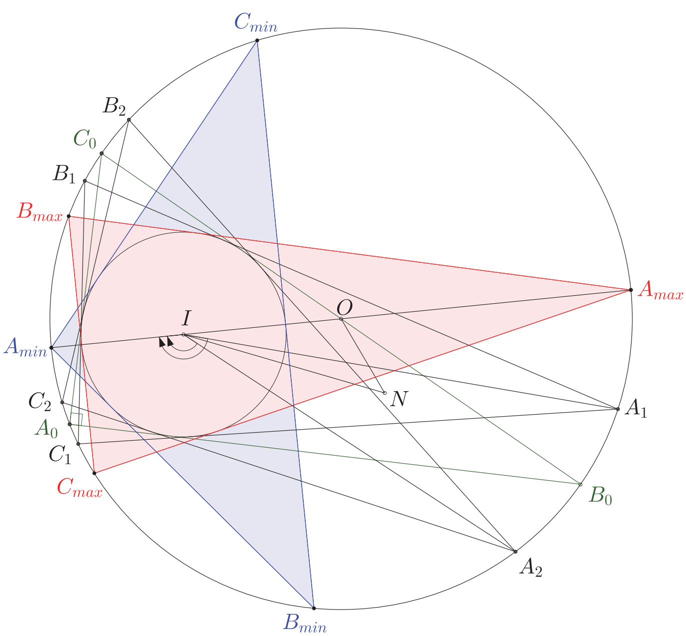

Figure 2. The distribution of triangles in the family $\mathscr{T}(R,r)$

On the other hand, from the relations $bc = \frac{4rRs}{a}$ and $b + c = 2s - a$, it follows that b, c are the roots of the quadratic equation

$$x^2 - (2s-a)x + \frac{4rRs}{a} = 0,$$

that is

$$\frac{2s - a \pm \sqrt{4s^2 - 4as + a^2 - \frac{16rRs}{a}}}{2}.$$

The above computations show that a triangle in the family $\mathscr{T}(R,r)$ is perfectly determined up to a congruence by the angle A. In this way, we obtain the distribution of triangles in the family $\mathscr{T}(R,r)$ (see Figure 2).

COROLLARY 5. *The distribution of triangles in the family $\mathscr{T}(R,r)$ is in pairs $(\Delta ABC, \Delta A'B'C')$ such that triangles ABC and $A'B'C'$ are congruent and symmetric with respect to the diameter OI.*

COROLLARY 6. *In the Blundon's configuration, the function $A \mapsto BC$ is strictly increasing on the arc $\widehat{A_{\max}A_0}$, and strictly decreasing on the arc $\widehat{A_0A_{\min}}$, where A_0 is the point on the semicircle $\widehat{A_{\max}A_{\min}}$ such that $\angle B_0A_0C_0 = \frac{\pi}{2}$.*

THEOREM 7. *(The strong version of Blundon's inequality) In the Blundon's configuration, the function $A \mapsto s(A)$, is strictly decreasing on the arc $\widehat{A_{\max}B_{\min}}$, where $s(A)$ denotes the semiperimeter of triangle ABC, that is we have the inequalities*

$$s(A_{\max}) \geqslant s(A) \geqslant s(B_{\min}).$$

Proof. Clearly, $s(A_{\max}) = s_{\max}$, the semiperimeter of triangle $A_{\max}B_{\max}C_{\max}$, and $s(A_{\min}) = s_{\min}$, the semiperimeter of triangle $A_{\min}B_{\min}C_{\min}$. When A moves on the arc $\widehat{A_{\max}B_{\min}}$ from $A_{\max}$ to $B_{\min}$, the angle $\angle ION$ strictly decreases from π to 0, i.e the function $A \mapsto \angle ION$ is strictly decreasing. Assume that we have the order $A_{\max}, A_1, A_2, B_{\min}$. From formula (2) we obtain $s^2(A_1) > s^2(A_2)$, and the conclusion follows. □

The area K of a triangle ABC in the family $\mathscr{T}(R,r)$ is a function of angle A, and we have the formula $K = K(A) = rs(A)$, where $s(A)$ is given in (15). The following consequence of Theorem 7 is the strong version of the result in [12, Theorem 1].

COROLLARY 8. *In the Blundon's configuration, the function $A \mapsto K(A)$ is strictly decreasing on the arc $\widehat{A_{\max}B_{\min}}$, strictly increasing on the arc $\widehat{B_{\min}C_{\max}}$, and strictly decreasing on $\widehat{C_{\max}A_{\min}}$, where $K(A)$ denotes the area of triangle ABC.*

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