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The orthopole theorem in the Poincaré disc model of hyperbolic geometry

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Abstract. In this study we prove the orthopole theorem for a hyperbolic triangle.

1 Introduction

Hyperbolic geometry appeared in the first half of the 19^{th} century as an attempt to understand Euclid's axiomatic basis of geometry. It is also known as a type of non-euclidean geometry, being in many respects similar to euclidean geometry. Hyperbolic geometry includes similar concepts as distance and angle. Both these geometries have many results in common but many are different. Several useful models of hyperbolic geometry are studied in the literature as, for instance, the Poincaré disc and ball models, the Poincaré half-plane model, and the Beltrami-Klein disc and ball models [5] etc. Following [8] and [9] and earlier discoveries, the Beltrami-Klein model is also known as the Einstein relativistic velocity model. Here, in this study, we give hyperbolic version of the orthopole theorem in the Poincaré disc model. The well-known orthopole theorem states that if A', B', C' be the projections of the vertices A, B, C of a triangle ABC on a straight line d, the perpendiculars from A' on

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BC, from B' on CA, and from C' on AB are concurrent at a point called the orthopole of d for the triangle ABC [4]. This result has a simple statement but it is of great interes. We just mention here few different proofs given by R. Goormaghtigh [3], J. Neuberg [6], W. Gallaty [2]. We use in this study the Poincaré disc model.

We begin with the recall of some basic geometric notions and properties in the Poincaré disc. Let D denote the unit disc in the complex z-plane, i.e.

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

The most general Möbius transformation of D is

$$z \to e^{i\theta} \frac{z_0 + z}{1 + \overline{z_0} z} = e^{i\theta}(z_0 \oplus z),$$

which induces the Möbius addition $\oplus$ in D, allowing the Möbius transformation of the disc to be viewed as a Möbius left gyro-translation

$$z \to z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0} z}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $z, z_0 \in D$, and $\overline{z_0}$ is the complex conjugate of z_0. Let $Aut(D, \oplus)$ be the automorphism group of the groupoid $(D, \oplus)$. If we define

$$gyr : D \times D \to Aut(D, \oplus)$$

by the equation

$$gyr[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\overline{b}}{1 + \overline{a}b},$$

then the following properties of $\oplus$ can be easy verified using algebraic calculation:

$a \oplus b = gyr[a, b](b \oplus a)$,	gyrocommutative law
$a \oplus (b \oplus c) = (a \oplus b) \oplus gyr[a, b]c$,	left gyroassociative law
$(a \oplus b) \oplus c = a \oplus (b \oplus gyr[b, a]c)$,	right gyroassociative law
$gyr[a, b] = gyr[a \oplus b, b]$,	left loop property
$gyr[a, b] = gyr[a, b \oplus a]$,	right loop property

For more details, please see [7].

Definition 1 *The hyperbolic distance function in* D *is defined by the equation*

$$d(a, b) = |a \ominus b| = \left| \frac{a - b}{1 - \overline{a}b} \right|.$$

Here, $a \ominus b = a \oplus (-b)$, *for* $a, b \in D$.

Theorem 1 ***(The Möbius Hyperbolic Pythagorean Theorem)*** *Let* ABC *be a gyrotriangle in a Möbius gyrovector space* $(V_s, \oplus, \otimes)$, *with vertices* $A, B, C \in V_s$, *sides* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{V_s}$ *and side gyrolenghts* $a, b, c \in (-s, s)$, $\mathbf{a} = -B \oplus C$, $\mathbf{b} = -C \oplus A$, $\mathbf{c} = -A \oplus B$, $a = \|\mathbf{a}\|$, $b = \|\mathbf{b}\|$, $c = \|\mathbf{c}\|$ *and with gyroangles* α, β, *and* γ *at the vertices* A, B, *and* C. *If* $\gamma = \pi/2$, *then*

$$\frac{c^2}{s} = \frac{a^2}{s} \oplus \frac{b^2}{s}$$

(see [8, p. 290]).

For further details we refer to the recent book of A. Ungar [7].

Theorem 2 ***(Converse of Carnot's theorem for hyperbolic triangle)*** *Let* ABC *be a hyperbolic triangle in the Poincaré disc, whose vertices are the points* A, B *and* C *of the disc and whose sides (directed counterclockwise) are* $a = -B \oplus C, b = -C \oplus A$ *and* $c = -A \oplus B$. *Let the points* A', B' *and* C' *be located on the sides* a, b *and* c *of the hyperbolic triangle* ABC, *respectively. If the following holds*

$$\left|-A \oplus C'\right|^2 \ominus \left|-B \oplus C'\right|^2 \oplus \left|-B \oplus A'\right|^2 \ominus \left|-C \oplus A'\right|^2 \oplus \left|-C \oplus B'\right|^2 \ominus \left|-A \oplus B'\right|^2 = 0,$$

and two of the three perpendiculars to the sides of the hyperbolic triangle at the points A', B' *and* C' *are concurrent, then the three perpendiculars are concurrent (See [1]).*

2 Main results

In this section, we prove the orthopole theorem for a hyperbolic triangle.

Theorem 3 *Let* A', B', C' *be the projections of the vertices* A, B, C *of the gyrotriangle* ABC *on a straight gyroline* d. *If two of the three perpendiculars from* A' *on* BC, *from* B' *on* CA, *and from* C' *on* AB *are concurrent, then the three perpendiculars are concurrent.*

Proof. Let's note A'', B'', C'' the projections of the points A', B', C' on BC, CA, AB, respectively (See Figure 1).

If we use Theorem 1 in the gyrotriangles $AA'B'$ and $AA'C'$, we get

$$\left|-A \oplus B'\right|^2 = \left|-B' \oplus A'\right|^2 \oplus \left|-A' \oplus A\right|^2 \tag{1}$$

and

$$\left|-C' \oplus A\right|^2 = \left|-A \oplus A'\right|^2 \oplus \left|-A' \oplus C'\right|^2 \tag{2}$$

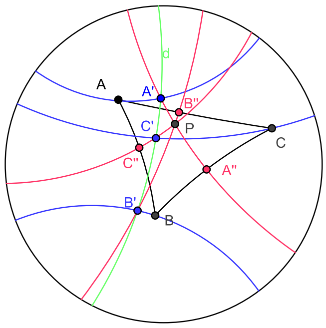

Figure 1

Figure 1: Projections of the points

Because $|-A' \oplus A|^2 = |-A \oplus A'|^2$, from the relations (1) and (2) we have

$$\left|-A \oplus B'\right|^2 \ominus \left|-B' \oplus A'\right|^2 = \left|-C' \oplus A\right|^2 \ominus \left|-A' \oplus C'\right|^2$$

i.e.

$$\alpha = \left|-A \oplus B'\right|^2 \ominus \left|-A \oplus C'\right|^2 = \left|-A' \oplus B'\right|^2 \ominus \left|-A' \oplus C'\right|^2 = \alpha' \tag{3}$$

Similary we prove that

$$\beta = \left|-B \oplus C'\right|^2 \ominus \left|-B \oplus A'\right|^2 = \left|-B' \oplus C'\right|^2 \ominus \left|-B' \oplus A'\right|^2 = \beta' \tag{4}$$

respectively

$$\gamma = \left|-C \oplus A'\right|^2 \ominus \left|-C \oplus B'\right|^2 = \left|-C' \oplus A'\right|^2 \ominus \left|-C' \oplus B'\right|^2 = \gamma'. \tag{5}$$

From the relations (3), (4) and (5) result

$$(\alpha \oplus \beta) \oplus \gamma = (\alpha' \oplus \beta') \oplus \gamma'.$$

Since $((-1, 1), \oplus)$ is a commutative group, we immediately obtain

$$\begin{aligned} &\left|-A \oplus B'\right|^2 \ominus \left|-A \oplus C'\right|^2 \oplus \left|-B \oplus C'\right|^2 \ominus \left|-B \oplus A'\right|^2 \\ &\oplus \left|-C \oplus A'\right|^2 \ominus \left|-C \oplus B'\right|^2 = 0. \end{aligned} \tag{6}$$

If we use the Theorem 1 in the gyrotriangles $AB'B''$, $AC'C''$, $BC'C''$, $BA'A''$, $CA'A''$ and $CB'B''$, we get

$$\left|-A \oplus B'\right|^2 = \left|-B' \oplus B''\right|^2 \oplus \left|-B'' \oplus A\right|^2, \tag{7}$$

$$\left|-A \oplus C'\right|^2 = \left|-C' \oplus C''\right|^2 \oplus \left|-C'' \oplus A\right|^2, \tag{8}$$

$$\left|-B \oplus C'\right|^2 = \left|-C' \oplus C''\right|^2 \oplus \left|-C'' \oplus B\right|^2, \tag{9}$$

$$\left|-B \oplus A'\right|^2 = \left|-A' \oplus A''\right|^2 \oplus \left|-A'' \oplus B\right|^2, \tag{10}$$

$$\left|-C \oplus A'\right|^2 = \left|-A' \oplus A''\right|^2 \oplus \left|-A'' \oplus C\right|^2, \tag{11}$$

$$\left|-C \oplus B'\right|^2 = \left|-B' \oplus B''\right|^2 \oplus \left|-B'' \oplus C\right|^2. \tag{12}$$

Now, from the relations (6) - (12), result

$$\left|-A \oplus B''\right|^2 \ominus \left|-A \oplus C''\right|^2 \oplus \left|-B \oplus C''\right|^2 \ominus \left|-B \oplus A''\right|^2 \oplus \left|-C \oplus A''\right|^2 \ominus \left|-C \oplus B''\right|^2 = 0,$$

and by Theorem 2 we obtain that the gyrolines $A'A'', B'B''$, and $C'C''$ are concurrent. □

Many of the theorems of Euclidean geometry have relatively similar form in the Poincare disc model, the orthopole theorem for a hyperbolic triangle is an example in this respect.

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