Mathematical
Inequalities
& Applications
[2462] First Galley Proofs

ANDRICA–IWATA'S INEQUALITY IN HYPERBOLIC TRIANGLE

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(*Communicated by V. Volenec*)

Abstract. In this paper we proof the Andrica-Iwata's inequality for a hyperbolic triangle.

1. Introduction

In the studies by Andrica in [1] and Iwata in [4], a basic theorem is established to be a sourse of inequalities from a euclidian triangle. Iwata's theorem states that if ABC is a triangle, and the segments BC, CA, AB have lengths $a, b, c,$ respectively, then

$$\frac{a}{b+c} \geqslant \sin\frac{A}{2}. \tag{1}$$

This result has a simple statement but it is of great interest. We just mention here few different proofs given by D. Mitrinović, J. Pečarić, V. Volenec [5], L. Balog [2], C. Ţiu [6]. In what follows we are going to present the counterpart of these results for the hyperbolic triangle.

THEOREM 1. (The Cosine Rule for Hyperbolic Triangle, see [3], p. 238). *Let ABC be a hyperbolic triangle, and the segments have hyperbolic lengths $d(B,C) = a$, $d(C,A) = b$, $d(A,B) = c$, then*

$$\sinh(b)\cdot\sinh(c)\cdot\cos(A) = \cosh(b)\cdot\cosh(c) - \cosh(a). \tag{2}$$

THEOREM 2. (The Sine Rule for Hyperbolic Triangle, see [3], p. 238). *Let ABC be a hyperbolic triangle, and the segments have hyperbolic lengths $d(B,C) = a$, $d(C,A) = b$, $d(A,B) = c$, then*

$$\frac{\sinh(a)}{\sin A} = \frac{\sinh(b)}{\sin B} = \frac{\sinh(c)}{\sin C}. \tag{3}$$

THEOREM 3. (The Hyperbolic Median Theorem, see [7]). *If AD is a median of the hyperbolic triangle ABC and the segments have hyperbolic lengths $d(B,C) = a$, $d(C,A) = b$, $d(A,B) = c$, $d(A,D) = d$, then*

$$\cosh(d) = \frac{\cosh(b)+\cosh(c)}{2\cosh\left(\frac{a}{2}\right)}. \tag{4}$$

Mathematics subject classification (2010): 30F45, 20N99, 51B10, 51M10.
Keywords and phrases: Hyperbolic inequality, hyperbolic triangle, Iwata's inequality.

2. Main results

In this section we proof the Iwata's inequality for a hyperbolic triangle.

THEOREM 4. *Let ABC be a hyperbolic acutetriangle or a right hyperbolic triangle in A, and the segments have hyperbolic lengths $d(B,C)=a,\ d(C,A)=b,\ d(A,B)=c$, then the following inequality holds*

$$\frac{\sinh(a)}{\sinh(b)+\sinh(c)}<\frac{1}{\sqrt{2}\cos\frac{\varepsilon+A}{2}},\tag{5}$$

where $\varepsilon=\pi-(A+B+C)$ is the defect of the triangle ABC.

Proof. If we use the sine rule (see Theorem 2) in the triangle ABC, we have

$$\frac{\sinh(a)}{\sinh(b)+\sinh(c)}=\frac{\sin A}{\sin B+\sin C}=\frac{\sin A}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}},\tag{6}$$

or

$$\frac{\sinh(a)}{\sinh(b)+\sinh(c)}=\frac{\sin A}{2\sin\frac{\pi-\varepsilon-A}{2}\cos\frac{B-C}{2}}=\frac{\sin A}{2\cos\frac{\varepsilon+A}{2}\cos\frac{B-C}{2}}.\tag{7}$$

Since $\sin A\leqslant 1$, we have

$$\frac{\sinh(a)}{\sinh(b)+\sinh(c)}\leqslant\frac{1}{2\cos\frac{\varepsilon+A}{2}\cos\frac{B-C}{2}}.\tag{8}$$

Because the hyperbolic triangle in acuteangle, it result that

$$|B-C|<\frac{\pi}{2}\tag{9}$$

i.e.

$$\left|\frac{B-C}{2}\right|<\frac{\pi}{4}.\tag{10}$$

But $\cos x$ is strictly decreasing on $(0,\pi/2)$, then

$$\cos\frac{B-C}{2}>\frac{\sqrt{2}}{2}.\tag{11}$$

By (8) and (11) we obtain the conclusion. □

COROLLARY 5. *Let ABC be a hyperbolic acutetriangle or a right hyperbolic triangle in A, and the segments have hyperbolic lengths $d(B,C)=a,\ d(C,A)=b$, $d(A,B)=c$, then the following inequality holds*

$$\frac{\sinh(a)}{\sinh(b)+\sinh(c)}<\frac{1}{\sqrt{2}\cos\frac{A}{2}}.\tag{12}$$

Proof. We have $\frac{\varepsilon+A}{2}=\frac{\pi}{2}-\frac{B+C}{2}$, and because $\cos x$ is strictly decreasing on $(0,\pi/2)$, then

$$\cos\frac{A}{2}>\cos\frac{\varepsilon+A}{2}. \tag{13}$$

By (5) and (13) we obtain the conclusion. □

COROLLARY 6. *Let ABC be a hyperbolic acutetriangle or a right hyperbolic triangle in A, and the segments have hyperbolic lengths $d(B,C)=a,\ d(C,A)=b$, $d(A,B)=c$, then the following inequality holds*

$$\sinh(a)<\sinh(b)+\sinh(c). \tag{14}$$

Proof. Using the fact that $\cos\frac{A}{2}>\frac{\sqrt{2}}{2}$ in the previous result we obtain

$$\frac{\sinh(a)}{\sinh(b)+\sinh(c)}<1,$$

and we are done. □

COROLLARY 7. *Let ABC be a hyperbolic acutetriangle and the segments have hyperbolic lengths $d(B,C)=a,\ d(C,A)=b,\ d(A,B)=c$, then the following inequality holds*

$$\frac{\sinh(a)}{\sinh(b)+\sinh(c)}<\frac{1}{\cos\frac{\varepsilon}{2}}. \tag{15}$$

Proof. From (5), we have

$$\frac{\sinh(a)}{\sinh(b)+\sinh(c)}<\frac{1}{\sqrt{2}\left(\cos\frac{\varepsilon}{2}\cos\frac{A}{2}-\sin\frac{\varepsilon}{2}\sin\frac{A}{2}\right)}, \tag{16}$$

hence

$$\frac{\sinh(a)}{\sinh(b)+\sinh(c)}<\frac{1}{\sqrt{2}\cos\frac{\varepsilon}{2}\cos\frac{A}{2}}. \tag{17}$$

Analogue with (11) we get

$$\frac{1}{\cos\frac{A}{2}}<\sqrt{2}. \tag{18}$$

By (17) and (18) we obtain the conclusion. □

COROLLARY 8. *Let ABC be a hyperbolic acutetriangle, and the segments have hyperbolic lengths $d(B,C)=a,\ d(C,A)=b,\ d(A,B)=c$, then*

$$\frac{\sinh(a)}{\sinh(b)+\sinh(c)}+\frac{\sinh(b)}{\sinh(a)+\sinh(c)}+\frac{\sinh(c)}{\sinh(b)+\sinh(a)}<\frac{3}{\cos\frac{\varepsilon}{2}}. \tag{19}$$

COROLLARY 9. *Let ABC be a hyperbolic triangle, and the segments have hyperbolic lengths* $d(B,C) = a,\ d(C,A) = b,\ d(A,B) = c,$ *then*

$$\frac{\sinh(a)}{\sinh(b)+\sinh(c)} + \frac{\sinh(b)}{\sinh(a)+\sinh(c)} + \frac{\sinh(c)}{\sinh(b)+\sinh(a)} \geqslant \frac{3}{2}. \tag{20}$$

Proof. If we use the inequality

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geqslant \frac{3}{2}, \tag{21}$$

where $x,y,z > 0,$ for the positive numbers $\sinh(a),$ $\sinh(b),$ and $\sinh(c),$ then we obtain the conclusion. □

REMARK 10. The equality

$$\frac{\sinh(a)}{\sinh(b)+\sinh(c)} + \frac{\sinh(b)}{\sinh(a)+\sinh(c)} + \frac{\sinh(c)}{\sinh(b)+\sinh(a)} = \frac{3}{2} \tag{22}$$

holds if and only if ABC is a equilateral triangle.

Proof. The relation (22) holds if and only if $\sinh(a) = \sinh(b) = \sin(c),$ so $a = b = c.$ □

THEOREM 11. *Let ABC be a hyperbolic triangle, and the segments have hyperbolic lengths* $d(B,C) = a,\ d(C,A) = b,\ d(A,B) = c,$ *then*

$$\cosh(b) + \cosh(c) > 2\cosh\left(\frac{a}{2}\right). \tag{23}$$

Proof. If AD is a median of the hyperbolic triangle $ABC,$ and $d(A,D) = d$ (see Figure 1), then from Theorem 3 we have

$$\cosh(d) = \frac{\cosh(b)+\cosh(c)}{2\cosh\left(\frac{a}{2}\right)}. \tag{24}$$

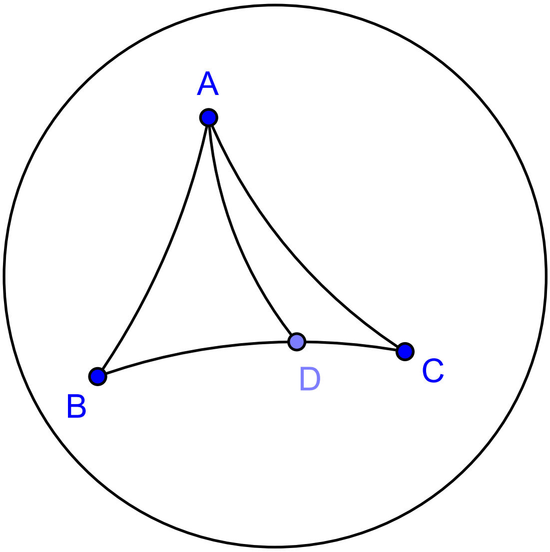

Figure 1

Figure 1.

Because the function $\cosh x > 1$, for all x, results

$$\frac{\cosh(b) + \cos(c)}{2\cosh\left(\frac{a}{2}\right)} > 1, \tag{25}$$

and the conclusion follows. $\square$

THEOREM 12. *If AD is a median of the hyperbolic triangle ABC and the segments have hyperbolic lengths $d(B,C) = a$, $d(C,A) = b$, $d(A,B) = c$, $d(A,D) = d$, then*

$$\sinh(d) > \frac{\cosh(b) - \cosh(c)}{2\sinh\left(\frac{a}{2}\right)}. \tag{26}$$

Proof. If we use the hyperbolic triangle inequality in the triangle ADC, we have $d + \frac{a}{2} > b$. Because the function $\cosh x$ is increasing on $(0,\infty)$, then

$$\cosh\left(d + \frac{a}{2}\right) > \cosh b, \tag{27}$$

or

$$\cosh(d) \cdot \cosh\left(\frac{a}{2}\right) + \sinh(d) \cdot \sinh\left(\frac{a}{2}\right) > \cosh b. \tag{28}$$

From the relations (4) and (28) we obtain

$$\frac{\cosh(b) + \cos(c)}{2} + \sinh(d) \cdot \sinh\left(\frac{a}{2}\right) > \cosh b, \tag{29}$$

the conclusion follows. □

COROLLARY 13. *If AD is a median of the hyperbolic triangle ABC and the segments have hyperbolic lengths $d(B,C) = a,\ d(C,A) = b,\ d(A,B) = c,\ d(A,D) = d,$ then*

$$2\sqrt{\sinh\frac{b+c}{2}\left|\sinh\frac{b-c}{2}\right|} - \sinh\left(\frac{a}{2}\right) < \sinh(d) < \frac{1}{\sqrt{2}\cos C}\left[\sinh\left(\frac{a}{2}\right) + \sinh(b)\right] \tag{30}$$

Proof. According to the inequality (12), for the triangle $ADC,$ we can write

$$\frac{\sinh(d)}{\sinh\left(\frac{a}{2}\right) + \sinh(b)} \leqslant \frac{1}{\sqrt{2}\cos C}, \tag{31}$$

and the right inequality in (32) is proved. For the left inequality to (30), we using the arithmetic mean-geometric mean inequality for the positive numbers $\sinh(d)$ and $\sinh\frac{a}{2}$ we get

$$2\sqrt{\sinh(d)\cdot\sinh\left(\frac{a}{2}\right)} \leqslant \sinh(d) + \sinh\left(\frac{a}{2}\right). \tag{32}$$

Now, we are using the inequalities (26) and (32), we get

$$\sqrt{2\left|\cosh(b) - \cosh(c)\right|} < \sinh(d) + \sinh\left(\frac{a}{2}\right). \tag{33}$$

Because

$$\cosh(b) - \cosh(c) = 2\sinh\frac{b+c}{2}\sinh\frac{b-c}{2}, \tag{34}$$

from (33) result

$$2\sqrt{\sinh\frac{b+c}{2}\left|\sinh\frac{b-c}{2}\right|} < \sinh(d) + \sinh\left(\frac{a}{2}\right), \tag{35}$$

the conclusion follows. □

THEOREM 14. *Let ABC be a hyperbolic triangle, and the segments have hyperbolic lengths $d(B,C) = a,\ d(C,A) = b,\ d(A,B) = c,$ then*

$$\sinh(a) \geqslant \sqrt{\cosh(a) - \cosh(b-c)} \tag{36}$$

Proof. Inequality (36) is equivalent with

$$\sinh(a) \geqslant \sqrt{\sinh(b)\cdot\sinh(c) - \cosh(b)\cdot\cosh(c) + \cosh(a)} \tag{37}$$

i.e.

$$\sinh^2(a) \geqslant \sinh(b)\cdot\sinh(c) - \cosh(b)\cdot\cosh(c) + \cosh(a) \tag{38}$$

From (2) and (38) we get

$$\begin{aligned} \sinh^2(a) \geqslant{} & \sinh(b)\cdot\sinh(c) - \cosh(b)\cdot\cosh(c) \\ & + \cosh(b)\cdot\cosh(c) - \sinh(b)\cdot\sinh(c)\cdot\cos A \end{aligned} \tag{39}$$

or

$$\sinh^2(a) \geqslant \sinh(b)\cdot\sinh(c)\,(1-\cos A)\,. \tag{40}$$

This inequality is equivalent to

$$\frac{\sinh(a)}{\sinh(b)}\cdot\frac{\sinh(a)}{\sinh(c)} \geqslant 1-\cos(A). \tag{41}$$

If we use the hyperbolic law of sines (see Theorem 2), inequality (41) becomes

$$\frac{\sin A}{\sin B}\cdot\frac{\sin A}{\sin C} \geqslant 1-\cos A \tag{42}$$

or

$$1-\cos^2 A \geqslant \sin B\cdot\sin C\cdot(1-\cos A). \tag{43}$$

i.e.

$$(1-\cos A)(1+\cos A) \geqslant \sin B\cdot\sin C\cdot(1-\cos A), \tag{44}$$

and it follows that

$$1+\cos A \geqslant \sin B\cdot\sin C, \tag{45}$$

and we are done. □

REMARK 15. Using the formula

$$\cosh(b)-\cosh(c) = 2\sinh\left(\frac{b+c}{2}\right)\sinh\left(\frac{b-c}{2}\right), \tag{46}$$

in the previous result we can write

$$\sinh(a) \geqslant \sqrt{2\sinh\left(\frac{a+b-c}{2}\right)\sinh\left(\frac{a+c-b}{2}\right)} \tag{47}$$

and the similar relation for $\sinh(b)$ and $\sinh(c)$.

COROLLARY 16. *Let ABC be a hyperbolic triangle and the segments have hyperbolic lengths $d(B,C)=a,\ d(C,A)=b,\ d(A,B)=c$, then the following inequalities hold*

$$2\sqrt{2}\prod_{cyclic}\sinh(s-a) \leqslant \prod_{cyclic}\sinh(a) < \frac{\prod_{cyclic}[\sinh(a)+\sinh(b)]}{2\sqrt{2}\prod_{cyclic}\cos\frac{A}{2}} \tag{48}$$

where s is the semiperimeter of the triangle ABC.

Proof. The left inequality results by (47), and the right inequality is a simple direct consequence of the relation (12). □

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(Received August 16, 2010)

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