Article

The χ-Hessian quotient for Riemannian metrics

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Abstract: Pseudo-Riemannian geometry and Hilbert-Schmidt norms are two important fields of research in applied mathematics. One of the main goals of this paper will be to find a link between these two research fields. In this respect, in the present paper we will introduce and analyze two important quantities in pseudo-Riemannian geometry, namely the H-distorsion and respectively the Hessian χ-quotient. This second quantity will be investigated using the Frobenius (Hilbert-Schmidt) norm. Some important examples will be also given which will proved the validity of the developed theory along the paper.

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1. Introduction

The Hessian structural geometry is a fascinating emerging area of research. It is in particular, related to Kaehlerian geometry, and also with many important pure mathematical fields of research, such as: affine differential geometry, cohomology and homogeneous spaces. A strong relationship can be also established with the geometry of information in applied mathematics. This systematic introduction to the subject initially develops the foundations of Hessian structures on the basis of a certain pair of a flat connection and a Riemannian metric, and then describes these related fields as theoretical applications.

In Finsler geometry, respectively in Riemannian geometry are few known invariants. So, one of the main path of research is to find new invariants and to study their impact in some concrete examples.
As we know B_x^n represents in Finsler geometry the unit ball in a Finsler space centered at $p \in M$, where (M,F) is a Finsler manifold, i.e.

$$B_x^n = \{y \in \mathbb{R}^n | |y| = F(x,y) < 1\}$$

and B^n represents the unit ball in the Euclidean space centered at origin:

$$B^n = \{x \in \mathbb{R}^n | |x| < 1\}$$

where $|x| = \sqrt{\delta_{ij}x^i x^j}$.

Also, in Finsler geometry we know that $\sigma_F(x)$ is given by:

$$\sigma_F(x) = \frac{volume(B^n)}{volume(B_x^n)}.$$

Let now recall one classical definition

Definition 1.1. *([5]) For a Finsler space, the distorsion $\tau = \tau(x,y)$ is given by:*

$$\tau(x,y) = \ln\left(\frac{\sqrt{det(g_{ij})}}{\sigma_F(x)}\right).$$

Acording also to Shen ([5]), when $F = \sqrt{g_{ij}(x)y^i y^j}$, is Riemannian, then

$$\sigma_F(x) = \sqrt{det(g_{ij}(x))}.$$

Definition 1.2. *([4]) A pseudo-Riemannian metric of metric signature (p,q) on a smooth manifold M of dimension $n = p+q$, is a smooth symmetric differentiable 2-form g on M, such that, at each point $x \in M$, g_x is non-degenerate on T_xM with the signature (p,q). We call (M,g) a pseudo-Riemannian manifold.*

Also a well known results from [4], is the following:

Theorem 1.1. *([[4]]) Given a pseudo-Riemannian manifold (M,g), there exists a unique linear connection ∇_g on M, called the Levi-Civita connection of g, such that:*

(a) $\nabla_g g = 0$,

(b) ∇_g *is torsion free, i.e.,* $T = 0$.

For a coordinate chart $(U, x^1, \cdots, x^n)$, the Christoffel symbols Γ^k_{ij} of the Levi-Civita connection are related to the components of the metric g in the following way:

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(\frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right).$$

If $f : M \to \mathbb{R}$ is a smooth function, then the second covariant derivative of the function f is given by:

$$\nabla^2_g f = \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial f}{\partial x^k}\right) dx^i \otimes dx^j$$

is called the Hessian of the function f.

In the paper [2], the authors have used the following notations, that we will also use in this paper:

$$\begin{aligned} f_{,i} &= \frac{\partial f}{\partial x^i}, \\ f_{,ij} &= \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij} f_{,m} \\ f_{,ijk} &= \frac{\partial f_{,ij}}{\partial x^k} - \Gamma^l_{ki} f_{,lj} - \Gamma^l_{kj} f_{,li}. \end{aligned}$$

Some important results regarding the "size" of a matrix were established in a series of papers recently. In this respect, please see [6], [1].

As we know, the two norm for a matrix A is given by

$$\|A\|_2 = max_{\|x\|=1}\|Ax\| = max_{x\neq 0}\frac{\|Ax\|}{\|x\|}.$$

The Frobenius (or the Hilbert-Schmidt norm) of a matrix $A = (A_{ij})$ is defined as follows:

$$\|A\|_{HS} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}^2}.$$

The operator norm of a matrix $A = (A_{ij})$ is given by

$$\|A\|_{op} = max_{\|x\|=1}\|Ax\|.$$

Next we will recall some properties of these norms. We will be focused on Hilbert-Schmidt norm $\|\cdot\|_{HS}$ because we will use it to established some new main results of this paper:

$$\|A \cdot B\|_{HS} \leq \|A\|_{HS} \cdot \|B\|_{HS} \tag{1.1}$$

$$\|A \cdot B\|_{op} \leq \|A\|_{op} \cdot \|B\|_{op} \tag{1.2}$$

$$\|A\|_{HS} \leq \sqrt{n}\|A\|_{op}. \tag{1.3}$$

In the previous inequality, the equality take place when $A = I_n$. Also,

$$|det(A)| \leq \|A\|_{HS}^n \tag{1.4}$$

$$\|A\|_{HS} \leq \sqrt{r}\|A\|_2 = \frac{\sqrt{r}}{\sigma_{min}(A)}. \tag{1.5}$$

Here, A denotes any positive definite symmetric matrix, r is the rank of A, and $\sigma_{min}(A)$ denotes the minimum singular value of A. Some interesting results regarding the Minkowski norm on Finsler geometry are presented in [3].

2. Main Results

Now, using the distortion definition we will introduce the following definition:

Definition 2.1. *For a pseudo-Riemannian manifold, we will denote by*

$$\sigma_{F,\nabla F} = \frac{\sigma_F(x)}{\sigma_{\nabla F}(x)} = \sqrt{\frac{det(g_{ij})}{det\left(\nabla_g^2 f\right)}} \tag{2.1}$$

the H-distortion, if and only if $\sigma_{F,\nabla F}^2(x) =$*constant.*

Example 2.1. *We will consider a pseudo-Riemannian manifold,* $\left\{(x^1, x^2)\,|\,x^1 \neq -1\right\} \subset \mathbb{R}^2$*, endowed with the following metric:*

$$g_{ij} = \begin{pmatrix} (1+x^1)^2 & x^1+2 \\ x^1 & 1 \end{pmatrix}. \tag{2.2}$$

After tedious computations, we get the following Christoffel symbols for this metric: $\Gamma^1_{11} = -1$, $\Gamma^2_{11} = x^1 + 1$, $\Gamma^2_{22} = \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{21} = \Gamma^2_{12} = \Gamma^1_{22} = 0$. *Now, we get:*

$$\begin{aligned} f_{,11} =& \frac{\partial^2 f}{\partial x^1 \partial x^1} - \left(\Gamma^1_{11} f_{,1} + \Gamma^2_{11} f_{,2}\right) \\ f_{,12} =& \frac{\partial^2 f}{\partial x^1 \partial x^2} - \left(\Gamma^1_{12} f_{,1} + \Gamma^2_{12} f_{,2}\right) \\ f_{,21} =& \frac{\partial^2 f}{\partial x^2 \partial x^1} - \left(\Gamma^1_{21} f_{,1} + \Gamma^2_{21} f_{,2}\right) \\ f_{,22} =& \frac{\partial^2 f}{\partial x^2 \partial x^2} - \left(\Gamma^1_{21} f_{,1} + \Gamma^2_{22} f_{,2}\right). \end{aligned}$$

With the above Christoffel symbols, we get after replacing, the following Hessian matrix:

$$\nabla^2_g f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^1 \partial x^1} + \frac{\partial f}{\partial x^1} - (x^1 + 1)\frac{\partial f}{\partial x^2} & \frac{\partial^2 f}{\partial x^1 \partial x^2} \\ \frac{\partial^2 f}{\partial x^2 \partial x^1} & \frac{\partial^2 f}{\partial x^2 \partial x^2} \end{pmatrix}.$$

We will search a function which respect the condition that the determinant of the Hessian will be a constant. In this respect, let us consider the following function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x^1, x^2) = kx^1x^2 + 1$, where k is a non-positive real constant. For this function, at a critical point x, the above Hessian matrix became:

$$\nabla^2_g f = Hf(x) = \begin{pmatrix} kx^2 + (x^1 + 1)kx^1 & k \\ k & 0 \end{pmatrix}.$$

Then the determinant of the Hessian will be $det(\nabla^2_g f) = -k^2 = constant$. Finally, let us conclude that in this case, the H-distorsion will be:

$$\sigma^2_{F,\nabla F} = \frac{det(g_{ij})}{det(\nabla^2_g f)} = \frac{-1}{k^2} = constant.$$

Now using the theory of Frobenius norms, we will introduce the following quantity:

Definition 2.2. *For a pseudo-Riemannian manifold* (M, g)*, we will denote by*

$$\chi_H = \frac{\|Hf(x)\|_{HS}}{\|Hf_1(x_1)\|_{HS}}, \tag{2.3}$$

the Hessian χ*-quotient for two smooth function* $f, f_1 : M \to \mathbb{R}$*. Here* $\|\cdot\|_{HS}$ *represent the Frobenius (Hilbert-Schmidt) norm of a Hessian matrix attached to the pseudo-Riemannian manifold. Here the point* x *represent the critical point for the first Hessian of the smooth function* f *and the point* x_1 *represent the critical point of the Hessian of the second function* f_1.

Remark 2.1. *Because* $\|Hf(x)\|_{HS}$ *and* $\|Hf_1(x_1)\|_{HS}$ *are two constants, then we can conclude that* χ_H *must be also a constant. Next we will investigate some of* χ_H *properties.*

First, let us recall the following well known properties about Frobenius norms: For two matrices $A = (a_{ij})$, respectively $B = (b_{ij})$, as we know, the following inequalities hold:

$$\|A \cdot B\|_{HS} \leq \|A\|_{HS} \cdot \|B\|_{HS}$$

respectively

$$\|B^{-1}\|_{HS} \leq \sqrt{r}\|B^{-1}\|_{HS}.$$

So, now we can formulate the following:

Theorem 2.1. *For the Hessian χ_H quotient for two smooth functions $f, f_1 : M \to \mathbb{R}$, considered for the same pseudo-Riemannian manifold (M, g), the following inequality holds:*

$$\chi_H \leq \frac{Q\sqrt{r}}{\sigma_{min}(Hf_1(x_1))}, \tag{2.4}$$

where $Q = \|Hf(x)\|_{HS}$, x is a critical point of the Hessian $Hf(x)$, r represent the rank of the second Hessian $Hf_1(x_1)$, x_1 is the critical point of the Hessian $Hf_1(x_1)$ and $\sigma_{min}\big(Hf_1^{-1}(x)\big)$ is the minimum singular value of the Hessian $Hf_1^{-1}(x)$.

Proof. Starting with the inequality

$$\|A \cdot B\|_{HS} \leq \|A\|_{HS} \cdot \|B\|_{HS},$$

for $A = Hf(x)$ and $B = (Hf_1(x_1))^{-1}$, one obtains:

$$\chi_H = \left\|Hf(x) \cdot Hf_1^{-1}(x_1)\right\|_{HS} \leq \|Hf(x)\|_{HS} \cdot \left\|Hf_1^{-1}(x_1)\right\|_{HS}.$$

Using the hypothesis of the theorem, we get from here, the following inequality:

$$\chi_H \leq Q \cdot \left\|Hf_1^{-1}(x_1)\right\|_{HS}.$$

Also we can remark the following fact:

$$\begin{aligned}\left\|Hf_1^{-1}(x_1)\right\|_{HS} &= \sqrt{tr\big(Hf_1^{-1}(x_1) \times Hf_1^{-1}(x_1)\big)} \\ &= \sum_{i=1}^{n} \sqrt{\frac{1}{\lambda_i^2}} \leq \sum_{i=1}^{n} \frac{1}{\lambda_i} = tr\big(Hf_1^{-1}(x_1)\big).\end{aligned}$$

But, using the inequalities between Frobenius norm and of the 2-norm, we get:

$$\left\|Hf_1^{-1}(x_1)\right\| \leq \sqrt{r} \cdot \left\|Hf_1^{-1}(x_1)\right\|_2$$

and since, $\left\|Hf_1^{-1}(x_1)\right\|_2 = \frac{1}{\sigma_{min}\left(Hf_1^{-1}(x_1)\right)}$, we find the desired result immediately. □

Remark 2.2. *Let us observe that, because*

$$\left\|Hf_1^{-1}(x_1)\right\|_{HS} \geq \left\|Hf_1^{-1}(x_1)\right\|_{op},$$

where $\|\cdot\|_{op}$ is the operator norm of the Hessian $Hf_1^{-1}(x_1)$, we obtain:
$\left\|Hf(x) \cdot Hf_1^{-1}(x_1)\right\|_{op} \leq \chi_H$.

Also because the general property

$$\|A\|_2 \leq \|A\|_{HS} \leq \sqrt{r}\|A\|_2$$

holds, we can give now the following:

Corollary 2.1. *For the Hessian χ_H quotient for two smooth functions $f, f_1 : M \to \mathbb{R}$, considered for the same pseudo-Riemannian manifold (M, g), the following inequality holds:*

$$\left\|Hf(x)Hf_1^{-1}(x_1)\right\|_2 \leq \chi_H \leq \sqrt{r}\left\|Hf(x)Hf_1^{-1}(x_1)\right\|_2, \tag{2.5}$$

where x is a critical point of the Hessian $Hf(x)$, r represent the rank of the second Hessian $Hf_1(x_1)$, x_1 is the critical point of the Hessian $Hf_1(x_1)$ and $\sigma_{min}\big(Hf_1^{-1}(x)\big)$ is the minimum singular value of the Hessian $Hf_1^{-1}(x)$.

Let now recall, from the first example where we get the Hessian for the function $f:\mathbb{R}^2 \to \mathbb{R}, f(x^1,x^2) = kx^1x^2+1$:

$$Hf(x) = \nabla_g^2 f(x) = \begin{pmatrix} (1+x^1)kx^1+kx^2 & k \\ k & 0 \end{pmatrix},$$

where $x = (x^1,x^2)$ is one critical point for the Hessian $Hf(x)$.

For the same metric as in Example 2.1,

$$g_{ij} = \begin{pmatrix} (1+x^1)^2 & x^1+2 \\ x^1 & 1 \end{pmatrix},$$

we will construct another function $f_1:\mathbb{R}^2 \to \mathbb{R}$, such that in a critical point $x_1 \in \mathbb{R}^2$, we could be able to compute the Hessian χ_H quotient. First, let us observe that for the metric (g_{ij}) from the first example, the Hessian of the function $f(x^1,x^2) = kx^1x^2+1$, has the critical point given as a solution of the bellow equation system:

$$\begin{cases} \frac{\partial f}{\partial x^1} = kx^2+1 = 0 \\ \frac{\partial f}{\partial x^1} = kx^1+1 = 0. \end{cases}$$

So, the critical point for this function will be $\left(-\frac{1}{k},-\frac{1}{k}\right)$. Next, the Hessian associated with the metric g_{ij} for this critical point will be:

$$Hf(x) = \nabla_g^2 f\left(-\frac{1}{k},-\frac{1}{k}\right) = \begin{pmatrix} -\frac{1}{k} & k \\ k & 0 \end{pmatrix}.$$

So, the Frobenius (Hilbert-Schmidt) norm in this case will be easily computed as follows:

$$\|Hf(x)_{HS}\| = \left\|\nabla_g^2 f\left(-\frac{1}{k},-\frac{1}{k}\right)\right\|_{HS} = \sqrt{2k^2+\left(\frac{1}{k}\right)^2}.$$

Let us now consider the following example as a continuation of the above results:

Example 2.2. *We will consider the following function $f_1:\mathbb{R}^2 \to \mathbb{R}$, $f(x^1,x^2) = 2x^1-(x^1)^2x^2+x^2$ for the same matrix as in the first example. The critical points for this function $f_1(x_1)$, are given as a solution of the following equations system*

$$\begin{cases} \frac{\partial f_1}{\partial x^1} = 2-2x^1x^2 = 0 \\ \frac{\partial f_1}{\partial x^2} = -(x^1)^2+1 = 0. \end{cases}$$

So, we obtain for the Hessian of this function f_1, two critical points $(1,1)$ respectively $(-1,-1)$. We will compute the Hessian matrix for this two critical points as follows:

$$\begin{aligned} Hf_1(x_1) = \nabla_g^2 f_1 =& \begin{pmatrix} -2x^2+2-2x^1x^2-(x^1+1)(1-x^1)^2 & -2x^1 \\ -2x^1 & 0 \end{pmatrix} \\ Hf_1(-1,-1) =& \nabla_g^2 f_1(-1,-1) = \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}. \end{aligned}$$

So, we derive $\|Hf_1(-1,-1)|\|_{HS} = \sqrt{2^2+2^2+2^2} = 2\sqrt{3}$.
In the same way, we point-out for the another critical point $(1,1)$ the following result for the Hessian matrix:

$$Hf_1(1,1) = \nabla_g^2 f_1(1,1) = \begin{pmatrix} -2 & -2 \\ -2 & 0 \end{pmatrix}.$$

So, we find $\|Hf_1(1,1)\|\|_{HS} = \sqrt{2^2+2^2+2^2} = 2\sqrt{3}$ *the same value as for the prevoius critical point. In conclusion, the Hessian* χ*-quotient for the above mentioned functions* $f, f_1 : \mathbb{R}^2 \to \mathbb{R}$*, is given by:*

$$\chi_H = \frac{\|Hf(x)\|_{HS}}{\|Hf_1(x_1)\|_{HS}} = \frac{\sqrt{2k^2 + \left(\frac{1}{k}\right)^2}}{2\sqrt{3}} = constant.$$

So, we have find an good example in respect with the definition that we have introduced earlier regarding the χ_H *quotient of some functions* f, f_1.

3. Conclusion

In this paper, we have investigated some interesting properties of the two quantities that we have introduced, namely, the H-distorsion and respectivelly the χ_H quotient. Finally, we have given some concludent examples regarding this theory.

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